# Exercise Sheet 5

To be handed in until October 25

## 1. Two Atlases on the Sphere

**(a)** Give an atlas of $2n+2$ charts on $S^n$ that are graphs.

**(b)** Give an atlas of 2 charts on $S^n$ that are given by stereographic projections.

## 2. An atlas on the real projective space

Let $\mathbb{RP}^n := \{\text{lines through the origin in } \mathbb{R}^{n+1}\}$. For $0 \neq x \in \mathbb{R}^{n+1}$, let $L = [x]$ be the line through $x$ and 0.

**(a)** Define a suitable topology on $\mathbb{RP}^n$. (Hint: Define a metric on $\mathbb{RP}^n$.)

**(b)** For any $j \in \{0, \dots n\}$ define

$$\begin{aligned} \mathbb{R}^n_j &:= \{x = (x^0, \dots, x^n) \in \mathbb{R}^{n+1} \,|\, x^j = 0\} \cong \mathbb{R}^n, \\ Z_j &:= \{L \in \mathbb{RP}^n \,|\, L \subset \mathbb{R}^n_j\}, \\ U_j &:= \mathbb{RP}^n \setminus Z_j. \end{aligned}$$

Show that $U_j = \{[x] \in \mathbb{RP}^n \,|\, x^j \neq 0\}$ and that $U_j$ is open in $\mathbb{RP}^n$.

**(c)** Define *homogeneous coordinates* on $U_j$ by $\psi_j : U_j \to \mathbb{R}^n$ by[1]

$$\psi_j([x]) = \frac{(x^0, \dots, \widehat{x^j}, \dots x^n)}{x^j}.$$

Prove that the maps $\psi_j$ are well-defined and that the coordinate systems $(U_j, \psi_j)$ give an atlas on $\mathbb{RP}^n$.

[1] The hat is a useful notation that means that this variable is omitted, i.e.

$$(x^0, \dots, \widehat{x^j}, \dots, x^n) := (x^0, \dots, x^{j-1}, x^{j+1}, \dots, x^n).$$

### 3. Two diffeomorphic but not equal structures on the real line

Consider $\mathbb{R}$ with its usual differentiable structure, induced by the chart $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$. Also consider the differentiable structure induced by the chart $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(x) = x^3$.

Show that the two differentiable structures are not equal, but that nevertheless, the two differentiable manifolds are diffeomorphic.

### 4. Quaternions

Let $Q$ denote the vector space $\mathbb{R}^4$ with basis $\{1, i, j, k\}$ and multiplication subject to the laws $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. (These make $Q$ into an *algebra*.)

**(a)** Show that every non–zero element $u \in Q$ is invertible.

Hint: Set $u = a + bi + cj + dk$. It is useful to define the *conjugate* $\bar{u} := a - bi - cj - dk$ and to prove $\bar{u}u = |u|^2 = u\bar{u}$.

**(b)** Show that $|uv| = |u||v|$ for $u, v \in Q$.

**(c)** Show that $S^3 := \{u \in Q \mid |u| = 1\}$ has the structure of a group.

### 5. For those new to topology

**(a)** Prove that the subspace topology is a topology.

**(b)** Prove that the quotient topology is a topology.

**(c)** Show that the subspace topology for $S^1$ in $\mathbb{R}^2$ coincides with the quotient topology $\mathbb{R} \to S^1$.

**(d)** Recall that a topological space $X$ is not connected if there are non-empty, open, disjoint subsets $U, V \subset X$ such that $U \cup V = X$. Any $U$ that arises this way (and $X$ itself) is called a component of $X$. Show that if $X$ is a manifold there is a unique decomposition

$$X = \bigcup_{\alpha} U_\alpha,$$

where $U_\alpha$ are disjoint connected components of $X$.

**(e)** What happens if we want to decompose the Cantor set $X$ as in (d)?