Exercise Sheet 6

To be handed in until November 01

1. A smooth projection

Prove: The projection $\pi : S^n \to \mathbb{RP}^n$, $x \mapsto [x]$ is smooth.

2. The orthogonal group

Let $O(n) := \left\{A \in \mathbb{R}^{n\times n} \mid A^T A = I\right\}$ be the group of orthogonal matrices. Characterize the tangent space $T_I O(n)$ of $O(n)$ at the identity I as follows.

(a) Let $A(t)$ be a smooth curve in $\mathbb{R}^{n\times n}$ with $A(0) = I$, $A(t) \in O(n)$. Find an equation satisfied by $B := dA(0)/dt$.

(b) The exponential map for sqaure matrices $C \in \mathbb{R}^{n\times n}$ is given by

$$e^C = \sum_{k=0}^{\infty} \frac{C^k}{k!}.$$

Prove that if $A(t) = e^{Bt}$ then $\frac{d}{dt}A(t) = Be^{Bt}$.

(c) For any B satisfying the equation from a) find a curve $A(t)$ in $O(n)$ with initial velocity B.

(d) A very beautiful picture of any such $A(t)$ comes by considering the diagonalization of orthogonal matrices to 2×2 blocks. Write $\mathbb{R}^n$ as the orthogonal sum of 2 dimensional subspaces V_i (and possible a 1-dimensional subspace W) and set each V_i rotating at constant angular speed θ_i.

(e) What is the dimension of $O(n)$?

3. Cutoff functions

Let M be a smooth manifold. For a function $u : M \to \mathbb{R}$ define the *support of* u as

$$\operatorname{spt}(u) := \overline{\{p \in M \,|\, u \neq 0\}}.$$

We say that a set Z is *compactly contained* in an open set $U \subset M$ if $\overline{Z}$ is compact and $\overline{Z} \subset U$. Write $K \subset\subset U$ in this case. Prove: if U is an open set in M and K a compact subset of U, then there exists a *cutoff function for* K *in* U, i.e. a function $\chi : M \to \mathbb{R}$ such that

i) χ is smooth,

ii) $0 \leq \chi \leq 1$,

iii) $\chi \equiv 1$ on K,

iv) $\operatorname{spt}(\chi) \subset\subset U$.

Hint: Recall without proof from analysis that the function

$$f(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

is a smooth map.

4. Coordinate vector fields are linearly independent

Prove that

$$\left(\frac{\partial}{\partial x^1}\right)_{p,\psi}, \dots, \left(\frac{\partial}{\partial x^n}\right)_{p,\psi}$$

are linearly independent tangent vectors.
Hint: Compute

$$\left(\frac{\partial}{\partial x^j}\right)_{p,\psi} \cdot (\zeta x^k),$$

where $x^1, \dots, x^n : U \to \mathbb{R}$ are the coordinate functions associated to (U, ψ) and ζ is a cutoff function for p in U (i.e. equal to 1 in a neighbourhood of p.)