Exercise Sheet 7

To be handed in until November 08

1. The Veronese embedding

(a) Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^4$ given by $F(x, y, z) := (x^2 - y^2, xy, xz, yz)$. Prove that F induces a well-defined map $f : \mathbb{RP}^2 \to \mathbb{R}^4$ characterized by $f([p]) := F(p)$ for any $p \in S^2$.

(b) Prove that f is injective.

(c) Prove that f is an immersion.

(d) Prove that f is a homeomorphism onto its image. (The map f is called the *Veronese embedding* of $\mathbb{RP}^2$ in $\mathbb{R}^4$. Note that $\mathbb{RP}^2$ does not embed in $\mathbb{R}^3$.)

2. TS^3 has a global trivialization

A *Lie group* is a smooth manifold endowed with a group structure such that the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth.

(a) Show that $S^3 := \{V \in Q : |V| = 1\}$ (where Q are the quaternions) is a Lie group.

(b) Construct smooth vector fields X, Y, Z on S^3 such that $X(u)$, $Y(u)$, $Z(u)$ are independent for each u in S^3. Conclude that $TS^3 \cong S^3 \times \mathbb{R}^3$.

3. The Hopf fibration

(a) Prove that every sphere of odd dimension carries a nowhere vanishing vector field.

(b) Prove that S^{2n-1} has a "smooth" decomposition into circles. They are called *Hopf fibers* of S^{2n-1}.

(c*) Can S^2 be decomposed into a disjoint union of submanifolds diffeomorphic to S^1?

4. Visualization of the Hopf fibration for S^3

Identify $\mathbb{C}^2$ with the quaternions Q by identifying $(z,w) = (a+bi, c+di) \in \mathbb{C}^2$ with $z + wj = a + bi + cj + dk \in Q$.

(a) Indentify $S^3 \setminus \{-1\}$ with $\mathbb{R}^3$ via stereographic projection from the point $-1 \in Q$. Locate in the target $\mathbb{R}^3$ the images of the points $1, \pm i, \pm j, \pm k$ and the 6 "coordinate" great circles of S^3.

(b) For $0 \leq r \leq \pi/2$, define

$$T_r := \{(z,w) : |z| = \cos(r), |w| = \sin(r)\}.$$

(i) Observe that $(T_r)_{0 \leq r \leq \pi/2}$ is a partition of S^3.

(ii) Observe that T_0 and $T_{\pi/2}$ are great circles of S^3 and are Hopf fibers in the sense of exercise 3.

(iii) Observe that for $0 < r < \pi/2$ the T_r are all tori, that they are equidistant from each other (in the path metric on S^3) and that each T_r is a union of Hopf fibers. The middle torus $T_{\pi/4} = \{(z,w) \,|\, |z| = |w| = 1/\sqrt{2})\}$ is called the *Clifford torus*.

(c) Visualize the Hopf fibration of S^3 by drawing all of the Hopf fibers in $\mathbb{R}^3$ (after stereographic projection). The tori T_r are useful guides.

(d) Remarkably, the quotient space $S^3/\sim$ is S^2, where $\sim$ is the equivalence relation where each Hopf fiber becomes a point. Can you "see" the S^2 that is swept out as the fiber S^1 varies in S^3? Can you find the upper and lower hemispheres of the S^2 in your diagram?