Exercise Sheet 8

To be handed in until November 15

1. Unit quaternions and rotations

Let $Q \cong \mathbb{R}^4$ be the quaternions and define the *purely imaginary quaternions* by

$$\mathbb{R}^3 := \{ai + bj + ck \in Q \,|\, a, b, c \in \mathbb{R}\} \cong \{0\} \times \mathbb{R}^3 \subset Q.$$

(a) Verify that the rule

$$Ad_v \colon w \mapsto vwv^{-1}$$

defines an action of the unit quaternions $v \in S^3$ on $\mathbb{R}^3$ by linear isometries (with respect to the usual inner product).

Hint: Use that for $u \in Q$ that u is purely imaginary iff $\overline{u} = -u$.

(b) For any quaternion $u \in Q$ define e^u by power series. Verify that for $n \in S^2 \subset \mathbb{R}^3$ and $\theta \in \mathbb{R}$ we have

$$e^{\theta n} = \cos\theta + n \sin\theta.$$

Show that $e^{\theta n} \in S^3$. Moreover, show that any v in $S^3 \subset Q$ can be written as $v = e^{\theta n}$ for some $\theta \in \mathbb{R}$ and $n \in S^2 \subset \mathbb{R}^3$ (i.e. the exponential $\mathbb{R}^3 \to S^3$ is surjective).

(c) Describe the action of an element v of S^3 on $\mathbb{R}^3$ *geometrically.*

Hint: Ad_v is a rotation by some angle ϕ about some axis. Find the axis and the angle.

(d) Verify that the association $v \mapsto Ad_v$ gives a surjective homomorphism and a two-sheeted covering map from S^3 to $SO(3)$. Consequently, observe that $SO(3) \cong \mathbb{R}P^3$.

2. Orientation and quotients

(a) Let M be a connected, oriented manifold, and suppose G is a group that acts freely and properly discontinuously on M by diffeomorphisms. Prove that M/G is orientable iff all $g \in G$ are orientation preserving.

(b) Show that $\mathbb{RP}^n$ is orientable iff n is odd.

3. Verseuchungsprinzip

Let M be a manifold and $U \subset M$ be open. Prove: If U is nonorientable then M is nonorientable.

4. Vector fields on the Klein bottle

Recall that the Klein bottle is $K = \mathbb{R}^2/G$ where G is the group generated by the maps

$$(x, y) \mapsto (x + 1, -y)$$
$$(x, y) \mapsto (x, y + 1).$$

How many pointwise linearly independent vector fields can **you** find on $\mathbb{R}^2/G$?

5. Orientation with curves

Let M be a smooth manifold.

(a) Let $p, q \in M$ and let $\gamma : [0, 1] \to M$ be a curve connecting p to q. Observe that any chosen orientation O of $T_{\gamma(0)}M$ propagates uniquely along γ to a unique path $O_\gamma(t)$ of orientations of $T_{\gamma(t)}M$ that is "continuous" in t (define this) and $O_\gamma(0)$.

(b) Let γ be a closed curve in M, i.e. $\gamma(0) = \gamma(1)$. We say that γ is *orientation-preserving* if $O_\gamma(0)$ equals $O_\gamma(1)$ (for any choice of $O_\gamma(0)$); otherwise we say that γ is *orientation-reversing*. Show that M is orientable if and only if every closed curve is orientation-preserving.

(c) Conclude that the Möbius strip and the Klein bottle are not orientable.

(d*) (For the ones that know cohomology): Define an element $w_1 \in H^1(M, \mathbb{Z}_2)$ that is measuring the obstruction of M being orientable, i.e. $w_1 = 0$ iff M is orientable. This w_1 is called *the first Stiefel-Whitney class*.