Exercise Sheet 9

To be handed in until November 22

1. Manifolds are locally compact in the strong sense (Lemma 0.5)

(a) Let M be a manifold. Show that for any open set $V \subset M$ and any point $p \in V$ there is an open set U such that

(i) $p \in U$,

(ii) $\overline{U}$ is compact,

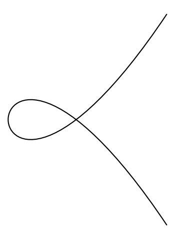
(iii) $\overline{U} \subset V$.

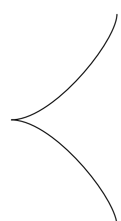
Note that this entails that M is locally compact but says more.

(b) Find a (non-Hausdorff) space X and $K \subset X$ such that K is compact but $\overline{K}$ is not compact.

2. Some non-submanifolds

Prove that the following curves are not submanifolds of $\mathbb{R}^2$:

(a)


(b)


3. Nonregular covering spaces

A covering space $\pi : M \to N$ is called *regular* if it comes from a group action, i.e. $N \cong M/G$ for some group G that acts freely and properly discontinuously on G. Find a nonregular covering space.

Hint: The base space N needs to have noncommutative fundamental group, e.g. consider the figure 8, or if you want a manifold consider either $\mathbb{C} \setminus \{\pm 1\}$ or the Klein bottle.

4. Another way to get the Hopf fibration

(a) Let US^2 denote the *unit tangent bundle* consisting of vectors in TS^2 with length 1 (in the usual metric). Show that US^2 is diffeomorphic to $SO(3)$.

(b) Show that the composition

$$S^3 \xrightarrow{Ad} SO(3) \cong US^2 \xrightarrow{\pi} S^2$$

is equivalent to the Hopf fibration from exercise sheet 7.

5. Equivalent definitions for sizes of the topology of manifolds

Prove that for a manifold M the following are equivalent:

(i) M is second countable.

(ii) M admits a countable atlas.

(iii) M is σ-compact.