# Exercise Sheet 1

To be handed in until September 27

**1. Lower bounds on the total absolute curvature**

**(a)** Prove that a closed regular curve $\gamma$ in $\mathbb{R}^2$ has

$$\int_\gamma |k|\, ds \geq 2\pi.$$

**(b)** Prove that a closed regular curve $\gamma$ in $\mathbb{R}^3$ has

$$\int_\gamma |k|\, ds \geq \pi.$$

**(c*)** (Fenchel's Theorem) Any closed regular curve $\gamma$ in $\mathbb{R}^3$ has

$$\int_\gamma |k|\, ds \geq 2\pi.$$

Moreover, there is an equality if and only if $\gamma$ is a plane convex curve.

**(d)** Recall (Milnor) that a knotted regular curve in $\mathbb{R}^3$ has $\int_\gamma |k|\, ds \geq 4\pi$. Prove: This bound is sharp, i.e. this cannot be improved to

$$\int_\gamma |k|\, ds \geq a$$

for any $a > 4\pi$.

**Solution:**

**(a)** We know from the lecture that $\int_\gamma k ds = 2\pi n$ for some $n \in \mathbb{Z}$. If $|n| \geq 1$ we get

$$\int_\gamma |k|\, ds \geq \left| \int_\gamma k ds \right| = |2\pi n| \geq 2\pi.$$

To include the case $n = 0$ we need another strategy. The unit tangent vector to a closed curve $\gamma : [c, d] \to \mathbb{R}^2$ is a function $\tau : [c, d] \to S^1 \subset \mathbb{R}^2$ with $\tau(c) = \tau(d)$.

**Claim.** The image of $\tau$ cannot be contained in an arc in $S^1$ of length strictly smaller than $\pi$ (that is, in an open half circle).

*Proof.* Suppose the image of $\tau$ is contained in an open half circle. Let's say (by rotating the plane) that the second coordinate of $\tau(t)$ in $\mathbb{R}^2$ is $> 0$ for all $t$. Then as $\tau$ is the derivative of $\gamma$ the second coordinate of $\gamma(t)$ in $\mathbb{R}^2$ is strictly increasing in $t$, so $\gamma(c) = \gamma(d)$ is not possible, which contradicts $\gamma$ being a closed curve. □

By the claim, the image of $\tau$ contains an arc of length $\pi$ because the image of $\tau$ is connected. But since $\tau$ is a closed curve in $S^1$ it must have length of at least $2\pi$. If $\tau$ is parametrized by arc length then the total curvature is exactly the length of the curve $\tau$. So

$$\int_\gamma |k|\, ds = \int_\gamma \left|\frac{\tau}{ds}\right| ds = \text{lenghth}(\tau) \geq 2\pi.$$

**(b)** Suppose $\gamma : [0, L] \to \mathbb{R}^3$ is parametrized by arc length. Let $\tau(s) = \frac{d\gamma}{ds}$ be the unit tangent vector and $\kappa(s) = \frac{d\gamma}{ds}$ the curvature vector. Note that $\tau$ is a function $[0, L] \to S^2$, hence a closed curve in $S^2$. Let $d_{S^2} : S^2 \times S^2 \to [0, \infty)$ be the distance function on $S^2$. It is given by the shortest path on the sphere between two points, i.e. the length of the shortest path on a great circle. This is exactly the angle between the two points. Therefore the formula

$$\cos(d_{S^2}(u, v)) = \langle u, v\rangle$$

holds for all $u, v \in S^2$, where $\langle\cdot, \cdot\rangle$ is the scalar product in $\mathbb{R}^3$.

**Claim.** For any $s \in [0, L]$ there is a $t \in [0, 1]$ such that

$$d_{S^2}(\tau(s), \tau(t)) \geq \frac{\pi}{2}.$$

*Proof.* Suppose

$$d_{S^2}(\tau(s), \tau(t)) < \frac{\pi}{2}$$

for any $t \in S^2$. Equivalently, by the formula for the distance above:

$$\langle\tau(s), \tau(t)\rangle > 0$$

for all $t \in [0, L]$. Without loss of generality (by rotating $\mathbb{R}^3$), assume $\tau(s) = (0, 0, 1)$ is the north pole in $S^2 \subset \mathbb{R}^3$. Set $h : [0, L] \to \mathbb{R}$ to be $\langle\gamma(s), (0, 0, 1)\rangle$, i.e. the third coordinate of $\tau(s)$. The negation of the condition in the claim would be that the last coordinate of $\tau(s)$ is positive for all $s$. This cannot be. More explicitly, since $\gamma(0) = \gamma(L)$ we have

$$0 = h(L) - h(0) = \int_0^L \frac{dh}{ds} ds = \int_0^L \left\langle \frac{d\gamma}{ds}, (0, 0, 1)\right\rangle ds > 0$$

which is a contradiction. □

Let $t \in [0, L]$ be such that the claim is satisfied for $s = 0$. Then

$$\int_\gamma |k|\, ds = \int_0^t \left|\frac{d\tau}{ds}\right| ds + \int_t^L \left|\frac{d\tau}{ds}\right| ds \geq d_{S^2}(\tau(0), \tau(t)) + d_{S^2}(\tau(t), \tau(0)) \geq \pi$$

**(c)** Fenchel's original proof (1929): `https://link.springer.com/article/10.1007/BF01454836`.
A shorter proof by Horn (1971): `https://www.tandfonline.com/doi/abs/10.1080/00029890.1971.11992766`. The main steps in the latter reference are the following.

(i) The curve $\tau$ has image in $S^2$ not contained in any open hemisphere. It is contained in a closed hemisphere iff $\gamma$ is a plane curve.

(ii) Any curve of length $\leq 2\pi$ in $S^2$ is contained in a closed hemisphere. Any curve of length $< 2\pi$ is strictly contained in an open hemisphere.

*Proof.* (i) The proof is similar to the proof of the claims in parts (a) and (b). Suppose the image of $\tau$ is contained in an open hemisphere. By rotating $\mathbb{R}^3$ we can assume that the last coordinate of $\tau(s)$ is $> 0$. Then as $\tau$ is the derivative of $\gamma$ the last coordinate of $\gamma(t)$ in $\mathbb{R}^3$ is strictly increasing in $t$, so $\gamma(0) = \gamma(L)$ is not possible, which contradicts $\gamma$ being a closed curve.

If the last coordinate of $\tau(s)$ is only $\geq 0$ for all $s$, then it actually must be $= 0$ for all $s$. Because if the last coordinate of $\tau(s)$ is $> 0$ for some $s$, there needs to be an $\tilde{s}$ such that the last coordinate of $\tau(s)$ is $< 0$ to get a closed curve (we need to lose height again). But if the last coordinate of $\tau(s)$ is 0 everywhere the curve stays in the plane with last coordinate 0.

(ii) Let $\tau$ be a closed curve of length $< 2\pi$ in $S^2$. We can divide the closed curve into two (not closed) curves (say $\tau_1, \tau_2$) of equal lengths at two points $P$ and $Q$. Let $N$ be the midpoint of the arc of the great circle on which $P$ and $Q$ lie. We show now that the curve $\tau$ lies in the hemisphere with north pole $N$. Suppose not, i.e. one of the curves, say $\tau_1$, crosses the equator with respect to the north pole $N$. Let $A$ be this intersection point. Then the curve $\tau_1$ together with its rotation around the axis going through the $N$ by an angle $\pi$ is a closed (piecewise smooth) curve of same length as $\tau$ and contains two antipodal points on the sphere ($A$ and the rotated $A$). But a curve going through two antipodal points on a sphere must have length at least $2\pi$. Note that the original $\tau$ has the same length as the curve $\tau_1$ together with the rotated $\tau_1$.

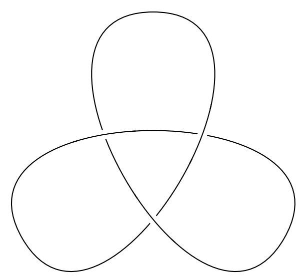
If $\tau$ is a closed curve of length $\leq 2\pi$, we can do the same argument as above to get that if $\tau$ crosses the equator, then $\tau$ contains two antipodal points, so must have length at least $2\pi$, hence the curve is exactly of length $2\pi$. But then $\tau$ needs to be a great circle because all other closed curves between two antipodal points have a strictly larger length than $2\pi$. Note now that a great circle lies obviously in a closed hemisphere.

□

The two claims imply the statement: If the length of $\tau$ (which is equal to $\int_\gamma |k| ds$) were strictly smaller than $2\pi$ then $\tau$ would be contained in an open hemisphere by (ii), which contradicts (i). So $\int_\gamma |k| ds \geq 2\pi$.

If the length of $\tau$ is exactly $2\pi$ then by (ii) lies in a closed hemisphere and by (1), $\gamma$ then is a plane curve.

**(d)** The trefoil as a curve with self-intersections in $\mathbb{R}^2$ has total absolute curvature $\int_\gamma |k|\, ds = 4\pi$.



If we start with a not embedded version of the trefoil in a plane $\mathbb{R}^2 \subset \mathbb{R}^3$ the trefoil still has total absolute curvature $4\pi$. To make the knot embedded in $\mathbb{R}^3$ we need to separate the 3 intersection points. This can be done by non-trivially curving one of the two segments that intersect close to an intersection point into an orthogonal direction to the plane $\mathbb{R}^2$. This creates non-trivial curvature in a new direction, where the curvature in $\mathbb{R}^2$ direction stays the same. The total curvature in the new direction can be made arbitrarily small (but nontrivial) to make space for the other segment and to get an embedded trefoil.

## 2. For those new to topology

**(a)** Let $X$ be a topological space. Show that if $X$ is path-connected, then $X$ is connected.

**(b)** Show that the continuous image of a connected topological space is connected.

**(c)** Let $U \subseteq \mathbb{R}^n$ be open set. Show that if $U$ is connected, then it is path-connected.

**(d)** Show that the continuous image of a compact topological space is compact.

**(e)** Let $X$ be a compact and Hausdorff topological space. Show that a subset $A \subset X$ is compact iff it is closed.

**Solution:**

**(a)** Suppose $X$ is not connected. Then there are nonempty disjoint open sets $U, V$ with $U \cup V = X$. Choose $x \in U$, $y \in V$. In case there is a continuous map $\gamma : [0,1] \to X$ with $\gamma(0) = x$ and $\gamma(1) = y$. Then $\gamma^{-1}(U)$ and $\gamma^{-1}(V)$ are two nonempty open disjoint sets covering $[0,1]$. But this contradicts $[0,1]$ being connected.

**(b)** By changing the target of the function $f$ from $Y$ to $f(X)$, the function stays continuous. So without loss of generality assume $f(X) = Y$. Suppose $Y$ is not connected. Then there are nonempty disjoint open sets $U, V$ in $Y$ with $U \cup V = Y$. But then $f^{-1}(U)$ and $f^{-1}(V)$ are two nonempty open disjoint sets covering $X$. But this contradicts $X$ being connected.

**(c)** Suppose $U$ is connected, nonempty and open. Choose $x \in U$. Define

$$V = \{y \in U \mid \text{ there is a path from } x \text{ to } y \text{ in } U\}.$$

The set $V$ is open in $U$: Let $y \in V$. There is an open ball centered at $y$ contained in $U$. Then there is also a path from $x$ to all elements in this ball by using the path from $x$ to $y$ and then going in a straight line to elements in the ball.

The set $V \setminus U$ is also open in $U$: If there is no path from $x$ to an element $y \in U$ there are also no paths from $x$ to elements of any small ball around $y$. Hence $V$ and and $U \setminus V$ are open, disjoint and cover $U$. As $x \in V$ (hence nonempty) and $U$ connected, $U \setminus V$ must be empty. This proves $V = U$ and hence $U$ path-connected.

**(d)** Let $f : X \to Y$ be a continuous function. Suppose $\mathcal{U} = (U_j)_{j \in J}$ is an open cover of a $f(X)$. By definition of the subspace topology of $Y$ there is for any $U_i \subset f(X)$ an open set $\tilde{U}_j \subset Y$ such that $U_j = \tilde{U}_j \cap f(X)$. By continuity of $f$, every set $f^{-1}(\tilde{U}_j) \subset X$ is open. Moreover, the family $\left(f^{-1}(\tilde{U}_j)\right)_{j \in J}$ is a cover of $X$. By compactness of $X$, there is a finite subcover $f^{-1}(\tilde{U}_1), \ldots, f^{-1}(\tilde{U}_m)$ also covering $X$. But then also $\tilde{U}_1, \ldots, \tilde{U}_m$ is a finite subcover covering $f(X)$, and so is $U_1, \ldots, U_m$.

**(e)** Suppose $A$ is closed. Let $\mathcal{U} = (U_j)_{j\in J}$ be an open cover of $A$. By definition of the subspace topology of $X$ there is for any $U_i \subset A$ an open set $\tilde{U}_j \subset X$ such that $U_j = \tilde{U}_j \cap A$. Then the family $(\tilde{U}_j)_{j\in J} \cup \{X \setminus A\}$ is an open cover of $X$. By compactness of $X$, there is a finite subcover $\tilde{U}_1, \dots, \tilde{U}_m$ and maybe $X \setminus A$. In any case $U_1, \dots, U_m$ is a finite open subcover of $A$.

Conversely, let $x \in X \setminus A$. We want to show that there is an open set $U$ in $X$ containing $x$ and being disjoint from $A$. Since $X$ is Hausdorff, for every $y \in A$ there are disjoint open sets $U_y$ and $V_y$ such that $x \in U_y$ and $y \in V_y$. Note that $(V_y)_{y\in A}$ is an open cover of $A$, so has a finite subcover, say $V_1, \dots, V_m$. Define $U = U_1 \cap \dots \cap U_m$. This set is open, contains $x$ and is disjoint from all the sets $V_1, \dots, V_m$ and hence also disjoint from $A$.