Exercise Sheet 10

To be handed in until November 29

1. Optimal embedding of product of spheres

What is the lowest dimension d such that $S^m \times S^n$ embeds into $\mathbb{R}^d$?

Solution:

We claim $d = n + m + 1$ is optimal. The sphere S^n embeds into $\mathbb{R}^{n+1}$ as the unit vectors. So $S^n \times S^m$ also embeds into $\mathbb{R}^{n+1} \times \mathbb{R}^{m+1} \cong \mathbb{R}^{n+m+2}$. Note that any $(x, y) \in S^n \times S^m$ has norm $\sqrt{2}$ as $|(x, y)|^2 = |x|^2 + |y|^2 = 2$. Hence $S^n \times S^m$ the map $i : S^n \times S^m \to S^{n+m+1} \subset \mathbb{R}^{n+m+2}$ given by

$$(x, y) \mapsto \frac{(x, y)}{\sqrt{2}}$$

is an embedding. The north pole N in S^{n+m+1} is not contained in the image $i(S^n \times S^m)$ so composing the map i with stereographic projection $S^{n+m+1} \setminus \{N\} \to \mathbb{R}^{n+m+1}$ produces an embedding

$$S^n \times S^m \to \mathbb{R}^{n+m+1}.$$

By exercise 2 there cannot be an embedding of the compact $(n+m)$-manifold $S^m \times S^n$ into $\mathbb{R}^{n+m}$, so we found the optimal dimension where $S^m \times S^n$ embeds.

2. Compact manifolds need at least one dimension higher to immerse

Prove that a compact n-dimensional manifold cannot be immersed into $\mathbb{R}^n$.

Solution:

Suppose $i : M^n \to \mathbb{R}^n$ is an immersion for M compact. Immersion means that $d_p i(p) : T_p M \to \mathbb{R}^n$ is an injective linear map for all $p \in M$. But for linear maps between vector spaces of the same dimension, injectivity, surjectivity and bijectivity are all equivalent. Hence $d_p i(p)$ is bijective for all p. But then by the inverse function theorem i is a local diffeomorphism. Local diffeomorphisms have open images: Around any $p \in M$ there is a small neighborhood U where i restricts to a diffeomorphism. So $i|_U : U \to i(U)$ is a diffeomorphism with $i(U) \subset \mathbb{R}^n$ an open neighbourhood of $i(p) \in \mathbb{R}^n$.

So $i(M) \subset \mathbb{R}^n$ is open. But as M is compact, also $i(M) \subset \mathbb{R}^n$ is compact. But the only open and compact subset of $\mathbb{R}^n$ is the empty set which is not possible to be $i(M)$.

3. More about the tangent space

(a) Prove $\pi : TM \to M$ is a submersion.

(b) Show that TM is always orientable (even if M is not).

Solution:

(a) Let $\psi : U \to \mathbb{R}^n$ be a chart. Then there is a chart $\Psi : TU \to \psi(U) \times \mathbb{R}^n$ of TM such that

$$\pi|_{TU} : TU \to U$$

is the projection $\psi(U) \times \mathbb{R}^n \to \mathbb{R}^n$ to the first coordinate using the coordinate charts Ψ and ψ. More concretely, if $\psi = (x^1, \dots, x^n) : U \to \mathbb{R}^n$ is the chart then $\Psi : TU \to \phi(U) \times \mathbb{R}^n$ is given by

$$\Psi = \left(x^1, \dots, x^n, \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right).$$

The projection $\phi(U) \times \mathbb{R}^n \to \phi(U)$ to the first coordinate map is a submersion. Hence also $\pi|_{TU} : TU \to U$ is a submersion. Being a submersion is a local property so also $\pi : TM \to M$ is a submersion.

(b) Let ψ, Ψ be as in part (a). For another chart $\phi : W \to \mathbb{R}^n$ denote $\Phi : TW \to \phi(W) \times \mathbb{R}^n$ the corresponding chart on TW. The transition function $\Psi \circ \Phi^{-1} : \phi(U \cap W) \times \mathbb{R}^n \to \psi(U \cap W) \times \mathbb{R}^n$ is given by

$$\Psi \circ \Phi^{-1}(x, v) = \Big(\psi \circ \phi^{-1}(x), D_x(\psi \circ \phi^{-1})(v) \Big)$$

for $x \in \psi(U \cap W)$ and $v \in \mathbb{R}$. The derivative of this map at some (x, v) in direction $(z, w) \in \mathbb{R}^n \times \mathbb{R}^n$ is given by

$$\begin{aligned}
D_{(x,v)}(\Psi \circ \Phi^{-1})(z, w) &= \frac{d}{dt}\Big|_{t=0} (\Psi \circ \Phi^{-1})(x + tz, v + tw) \\
&= \frac{d}{dt}\Big|_{t=0} \Big(\psi \circ \phi^{-1}(x + tz), D_{x+tz}(\psi \circ \phi^{-1})(v + tw) \Big) \\
&= \Big(D_x(\psi \circ \phi^{-1})(z), D_x^2(\psi \circ \phi^{-1})(v, z) + D_x(\psi \circ \phi^{-1})(w) \Big) \\
&= \begin{pmatrix} D_x(\psi \circ \phi^{-1}) & 0 \\ D_x^2(\psi \circ \phi^{-1})(v, \cdot) & D_x(\psi \circ \phi^{-1}) \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}.
\end{aligned}$$

So the determinant of the transition map is

$$\begin{aligned}
\det D_{(x,v)}(\Psi \circ \Phi^{-1}) &= \det \begin{pmatrix} D_x(\psi \circ \phi^{-1}) & 0 \\ D_x^2(\psi \circ \phi^{-1})(v, \cdot) & D_x(\psi \circ \phi^{-1}) \end{pmatrix} \\
&= \det D_x(\psi \circ \phi^{-1}) \det D_x(\psi \circ \phi^{-1}) \\
&= \big(\det D_x(\psi \circ \phi^{-1}) \big)^2 > 0.
\end{aligned}$$

This proves that there is an atlas on TM such that all transition maps have strictly positive determinants, which implies that TM is orientable.

4. Orthogonal and unitary matrices as submanifolds

(a) Prove that $O(n)$ and $SO(n)$ are compact submanifold of $\mathbb{R}^{n\times n}$. Prove that $O(n)$ has two connected components.

(b) Prove that

$$U(n) := \{A \in \mathbb{C}^{n\times n} \mid \overline{A^T}A = I\},$$
$$SU(n) := \{A \in U(n) \mid \det U = 1\}$$

are both compact submanifolds of $\mathbb{C}^{n\times n} \cong \mathbb{R}^{2n^2}$.

(c) Compute the tangent spaces $T_I U(n)$ and $T_I SU(n)$ at the identity I.

(d) Are $U(n)$ and $SU(n)$ connected?

Solution:

(a) The map $f : \mathbb{R}^{n\times n} \to \{\text{symmetric matrices}\}$ given by $A \mapsto A^T A$ is smooth. Note that the space of symmetric matrices is a vector space. The preimage under f of the identity $\{I\}$ is $O(n)$. As the one-elemented space $\{I\}$ is closed its preimage $O(n) = f^{-1}(\{I\})$ is closed. Moreover, as the pointwise (square of the) norm of a matrix A is $||A||^2 = \sum_{i=1}^n \sum_{j=1}^n A_{ij}^2$ is n (each column in A is a unit vector) the set $O(n)$ is bounded. Closed and bounded sets of $\mathbb{R}^{n\times n}$ are compact, hence $O(n)$ is compact.

To prove that the identity I is a regular value of f let us compute $D_A f : \mathbb{R}^{n\times n} \to \{\text{symmetric matrices}\}$. For $W \in \mathbb{R}^{n\times n}$ we have

$$\begin{aligned} D_A f(W) &= \frac{d}{dt}\Big|_{t=0} f(A + tW) \\ &= \frac{d}{dt}\Big|_{t=0} ((A + tW)^T (A + tW)) \\ &= \frac{d}{dt}\Big|_{t=0} (A^T A + t(A^T W + W^T A) + t^2 W^T W) \\ &= A^T W + W^T A. \end{aligned}$$

We need to show that for any $A \in O(n) = f^{-1}(\{I\})$ that $D_A f$ is surjective, i.e. that for any symmetric matrix B there is a matrix W such that $B = A^T W + W^T A$. This is true for $W = \frac{AB}{2}$ as $A^T A = I$ and $B^T = B$. This proves that $O(n) = f^{-1}(\{I\})$ is a manifold of dimension

$$\dim(\mathbb{R}^{n\times n}) - \dim(\text{symmetric matrices}) = n^2 - \frac{n(n+1)}{2} = \frac{n(n-1)}{2}.$$

Next, let us show that $O(n)$ has two connected components. We almost already proved that in exercise sheet 6 problem 2. Any $A \in SO(n)$ is conjugate to a block diagonal matrix

$$A = V\operatorname{diag}(R_{\theta_1}, \dots, R_{\theta_{n/2}})V^{-1}$$

with $\theta_1, \dots, \theta_{n/2} \in \mathbb{R}$, $V \in GL(n)$ and where $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$ a rotation matrix. This is true for n even. We only give the proof for n even because when n is odd there is an extra eigenvalue which is 1 and needs to be added to the diagonal matrix. The following argument is easily adjusted to odd n. One possible path from the identity to A is $[0,1] \to SO(n)$ given by

$$t \mapsto V\operatorname{diag}(R_{t\theta_1}, \dots, R_{t\theta_{n/2}})V^{-1}.$$

So $SO(n)$ is connected.

We claim that $O(n) = SO(n) \cup S(SO(n))$ where $S \in O(n)$ is any matrix with $\det S = -1$ (e.g. a reflection). Indeed, for $A \in O(n)$ we have $\det A \in \{\pm 1\}$. If $\det A = -1$ then $A = S(S^{-1})A) \in S(SO(n))$. Actually, $O(n) = SO(n) \cup S(SO(n))$ is the decomposition into connected components. The subset $S(SO(n))$ is connected as $SO(n)$ is connected and multiplying by a matrix is continuous. As a manifold $SO(n) \cong S(SO(n))$ (but not as Lie groups because the latter is not even a group). Because the determinant $\det : O(n) \to \{\pm 1\}$ is continuous, $O(n)$ must have at least two connected components as $\pm$

(b) The strategy of the proof is similar to (a). The map $f : \mathbb{C}^{n\times n} \to$ {hermitian matrices} given by $A \mapsto \overline{A^T}A$ is smooth. Note that the space of hermitian matrices is a real (not a complex!) vector space. The preimage under f of the identity $\{I\}$ is $U(n)$. As the one-elemented space $\{I\}$ is closed its preimage $U(n) = f^{-1}(\{I\})$ is closed. Moreover, as the pointwise (square of the) norm of a matrix A is $||A||^2 = \sum_{i=1}^n \sum_{j=1}^n |A_{ij}|^2$ is n (each column in A is a (complex) unit vector) the set $U(n)$ is bounded. Closed and bounded sets of $\mathbb{C}^{n\times n} \cong \mathbb{R}^{2n^2}$ are compact, hence $U(n)$ is compact.

To prove that the identity I is a regular value of f let us compute $D_A f : \mathbb{C}^{n\times n} \to$ {hermitian matrices}. For $W \in \mathbb{C}^{n\times n}$ we have

$$\begin{aligned} D_A f(W) &= \frac{d}{dt}\Big|_{t=0} f(A+tW) \\ &= \frac{d}{dt}\Big|_{t=0} (\overline{(A+tW)^T}(A+tW)) \\ &= \frac{d}{dt}\Big|_{t=0} (\overline{A^T}A + t(\overline{A^T}W + \overline{W^T}A) + t^2\overline{W^T}W) \\ &= \overline{A^T}W + \overline{W^T}A. \end{aligned}$$

We need to show that for any $A \in U(n) = f^{-1}(\{I\})$ that $D_A f$ is surjective, i.e. that for any hermitian matrix B there is a matrix W such that $B = \overline{A^T}W + \overline{W^T}A$. This is true for $W = \frac{AB}{2}$ as $\overline{A^T}A = I$ and $\overline{B^T} = B$. This proves that $U(n) = f^{-1}(\{I\})$ is a manifold of dimension

$$\dim(\mathbb{R}^{2n^2}) - \dim(\text{hermitian matrices}) = 2n^2 - n^2 = n^2.$$

Let us compute the derivative of $\det : \mathbb{C}^{n\times n} \to \mathbb{C}$ For $A = I$ and $W \in \mathbb{C}^{n\times n}$ we have

$$\begin{aligned} D_I \det(W) &= \frac{d}{dt}\Big|_{t=0} \det(I + tW) \\ &= \lim_{t\to 0} \frac{\det(I + tW) - \det I}{t} \\ &= \operatorname{tr}(W). \end{aligned}$$

We used that $\det(I + tW) = t^n \det(I/t - (-W)) = t^n \chi_{(-W)}(1/t)$ and the characteristic polynomial $\chi_W(\lambda) = \det(\lambda I - W)$ has as the λ^{n-1} coefficient the trace of $-W$.

Let now A be invertible and X any other matrix. Then taking the derivative of

$$\det X = \det(AA^{-1}X) = \det A \det(A^{-1}X)$$

with respect to X and evaluate it in direction W at $X = A$ we get

$$\begin{aligned} D_A \det(W) &= \det A\, D_{A^{-1}A} \det(A^{-1}W) \\ &= \det A\, D_I \det(A^{-1}W) \\ &= \det A \operatorname{tr}(A^{-1}W) \end{aligned}$$

using the chain rule since the derivative of $X \mapsto A^{-1}X$ is given by $W \mapsto A^{-1}W$ at any X.

For $A \in U(n)$ the determinant is a map $\det : U(n) \to S^1 \subset \mathbb{C}$ with derivative

$$D_A \det : T_A U(n) = \{W \in \mathbb{C}^{n\times n} \mid \overline{A^T}W + \overline{W^T}A = 0\} \to \mathbb{R}$$

given by

$$D_A \det(W) = \det A \operatorname{tr}(A^{-1}W).$$

The map $\det : U(n) \to S^1$ is a submersion: Because the target space is one-dimensional, surjectivity for the differential is equivalent to it being non-zero. Just choose a matrix $W \in T_A(U)$ such that $A^{-1}W$ has non-zero trace.

This proves that $SU(n) = \det^{-1}(\{1\})$ is a submanifold of $U(n)$ of dimension $\dim U(n) - \dim(S^1) = n^2 - 1$. Moreover, as $SU(n) = \det^{-1}(\{1\})$ it is a closed subset in $U(n)$ and as $U(n)$ is compact also $SU(n)$ is compact.

(c) The tangent space at $A \in U(n)$ is

$$T_A U(n) = \{W \in \mathbb{C}^{n\times n} \mid \overline{A^T} W + \overline{W^T} A = 0\}.$$

So the tangent space at the identity is

$$T_I U(n) = \{W \in \mathbb{C}^{n\times n} \mid W + \overline{W^T} = 0\},$$

i.e. the space of skew-hermitian matrices.

The tangent space at $A \in SU(n)$ is

$$T_A SU(n) = \{W \in \mathbb{C}^{n\times n} \mid \overline{A^T} W + \overline{W^T} A = 0,\ \det A \operatorname{tr}(A^{-1}W) = 0\}.$$

So the tangent space at the identity is

$$T_I U(n) = \{W \in \mathbb{C}^{n\times n} \mid W + \overline{W^T} = 0,\ \operatorname{tr}(W) = 0\},$$

i.e. the space of trace-zero skew-hermitian matrices.

(d) Let us first show that $U(n)$ is connected. Any unitary matrix $A \in U(n)$ can be diagonalized, i.e. there is an invertible matrix V and $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ such that

$$A = V \operatorname{diag}(\lambda_1, \dots, \lambda_n) V^{-1}.$$

Actually, $\lambda_j = e^{i\theta_j}$ for some $\theta_j \in \mathbb{R}$ as eigenvalues need to have norm 1. But then $[0,1] \to U(n)$ given by

$$t \mapsto V \operatorname{diag}(e^{it\theta_1}, \dots, e^{it\theta_n}) V^{-1}$$

is a path in $U(n)$ from the identity to A.

If $A \in SU(n)$ then $\theta_1 + \cdots + \theta_n = 0$ as $\det A = 1$. So the constructed path above in $U(n)$ is actually for all t in $SU(n)$ if A is.

5. The complex projective space

Let $\mathbb{CP}^n := \{\text{complex lines in } \mathbb{C}^{n+1} \text{ through the origin}\}$. Define the function $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ by

$$z = (z^0, \dots, z^n) \mapsto [z] = \{\lambda z \mid \lambda \in \mathbb{C}\} \in \mathbb{CP}^n.$$

(a) Find coordinate charts that make $\mathbb{CP}^n$ into a smooth $2n$-manifold.

(b) Observe that $\mathbb{CP}^1 \cong S^2$.

(c) Let $S^{2n+1} := \{z \in \mathbb{C}^{n+1} \mid |z| = 1\}$. The map $h : S^{2n+1} \to \mathbb{CP}^n$ given by $h(z) := [z]$ is called the *Hopf fibration*. Prove that h is a submersion. The fibers $h^{-1}(q)$, $q \in \mathbb{CP}^n$, yield a decomposition of S^{2n+1} into circles.

(d) Observe that in the case $n = 1$ we get the classical Hopf fibration

$$h : S^3 \to S^2$$

as defined in exercise sheet 7.

Solution:

(a) For $j = 0, \dots, n$ set $U_j = \{[z] \in \mathbb{CP}^n \mid z^j \neq 0\}$. This is well defined as if $[z] = [w]$ then $z^j \neq 0$ iff $w^j \neq 0$. Define charts $\psi_j : U_j \to \mathbb{C}^n \cong \mathbb{R}^{2n}$ given by

$$[z] \mapsto \frac{(z^0, \dots, \widehat{z^j}, \dots, z^n)}{z^j}.$$

This map is well-defined as for $[z] = [w]$ also $\psi_j([z]) = \psi_j([w])$ by the same argument as for the real projective space. The map is bijective with inverse $\psi_j^{-1} : \mathbb{C}^n \cong \mathbb{R}^{2n} \to U_j$

$$y \mapsto [(y^0, \dots, y^{j-1}, 1, y^j, \dots, y^n)].$$

We equip $\mathbb{CP}^n$ with the quotient topology coming from the map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ sending $z \mapsto [z]$. More concretely, a set $U \subset \mathbb{CP}^n$ is open iff the union of the complex lines in U as a subset of $\mathbb{C}^{n+1} \setminus \{0\}$ is open. With this topology on $\mathbb{CP}^n$, the map ψ_j is a homeomorphism. Indeed, for $U \subset \mathbb{C}^n$,

$$\pi^{-1}(\psi_j^{-1}(U)) = \{\lambda(y^0, \dots, y^{j-1}, 1, y^j, \dots, y^n) \mid \lambda \in \mathbb{C}\setminus\{0\}, y \in U\} \subset \mathbb{C}^{n+1}\setminus\{0\}.$$

This set is open in $\mathbb{C}^{n+1} \setminus \{0\}$ iff U is open in $\mathbb{C}^n$. That the charts are compatible is proved exactly as for the real projective space (see exercise 2 sheet 5).

(b) $\mathbb{CP}^1$ and S^2 are both the one-point compactification of the complex plane $\mathbb{C}$. The one-point compactification $\overline{\mathbb{C}}$ of the complex plane is as a set the disjoint union $\mathbb{C} \cup \{\infty\}$. The topology is the following. A subset $U \subset \overline{\mathbb{C}}$ that does not contain ∞ is open iff it is open in $\mathbb{C}$. A subset U in $\overline{\mathbb{C}}$ that contains ∞ is open iff its complement is compact in $\mathbb{C}$.

The map $S^2 \to \overline{\mathbb{C}}$ defined by stereographic projection on $S^2 \setminus \{N\} \to \mathbb{R}^2 \cong \mathbb{C} \subset \overline{\mathbb{C}}$ and sending the north pole $N \mapsto \infty$ is a homeomorphism.

Also the map $\mathbb{CP}^1 \to \overline{\mathbb{C}}$ defined by

$$[z^0, z^1] \mapsto \begin{cases} \frac{z^0}{z^1}, & z^1 \neq 0, \\ \infty, & z^1 = 0 \end{cases}$$

is a homeomorphism. On $U_1 \subset \mathbb{C}$ the map is simply the chart and $\mathbb{CP}^1 \setminus U_1$ is simply a point in $\mathbb{CP}^1$, the line $\{(\lambda, 0) \mid \lambda \in \mathbb{C}\}$.

To check that the composition $S^2 \to \overline{\mathbb{C}} \to \mathbb{CP}^1$ is smooth is a straightforward computation.

(c) For $k = 0, \ldots, 2n+1$ let $\psi_k^{\pm} : U_j^{\pm} \to \mathbb{R}^{2n+1}$ be the charts on the sphere S^{2n+1} defined in exercise 1 sheet 5. Then $h(U_{2j}^{\pm}) \subset U_j$ and $h(U_{2j+1}^{\pm}) \subset U_j$ for $j = 0, \ldots n$ as either the real or the complex part of a complex number needs to be non-zero for the complex number to be non-zero. The composition

$$B^{2n+1} \xrightarrow{(\psi_{2j}^{\pm})^{-1}} U_{2j}^{\pm} \xrightarrow{\pi} U_j \xrightarrow{\psi_j} \mathbb{C}^n \cong \mathbb{R}^{2n}$$

is given by

$$(y^0, \ldots, y^{2n}) \mapsto \frac{(y^0, \ldots, \widehat{y^j}, \ldots, y^{2n})}{\pm\sqrt{1-(y^j)^2} + iy^j}.$$

where we think of $(y_0, \ldots, \widehat{y^j}, \ldots, y^{2n}) \in \mathbb{R}^{2n} \cong \mathbb{C}^n$ as a tuple of complex numbers such that we can divide by a complex number. Similarly, the composition

$$B^{2n+1} \xrightarrow{(\psi_{2j+1}^{\pm})^{-1}} U_{2j+1}^{\pm} \xrightarrow{\pi} U_j \xrightarrow{\psi_j} \mathbb{C}^n \cong \mathbb{R}^{2n}$$

is given by

$$(y^0, \ldots, y^{2n}) \mapsto \frac{(y^0, \ldots, \widehat{y^j}, \ldots, y^{2n})}{y^j + i \pm \sqrt{1-(y^j)^2}}.$$

These maps are smooth, so h is differentiable. Moreover, the map is a submersion. We can already see this when taking the partial derivative of the map $f_j : B^{2n+1} \xrightarrow{(\psi_{2j}^{\pm})^{-1}} U_{2j}^{\pm} \xrightarrow{\pi} U_j \xrightarrow{\psi_j} \mathbb{C}^n \cong \mathbb{R}^{2n}$ with respect to all y_k except $k = j$ as we get

$$\frac{\partial f_j}{\partial y^k}(y) = \left(0, \ldots, 0, \frac{1}{y^j + i \pm \sqrt{1-(y^j)^2}}, 0, \ldots 0\right)$$

with the non-zero entry at position k.

If $z, w \in h^{-1}(q)$ then $z = \lambda w$ for $\lambda \in \mathbb{C} \setminus \{0\}$. Moreover, λ must have norm 1 as z and w do. So $z = e^{it}w$. We get Hopf fibers.

(d) We already showed in (c) that we get Hopf fibers.