Exercise Sheet 11

To be handed in until December 06

1. Some explicit computations of Lie brackets

Given $v \in \mathbb{R}^3$, define vector fields on $\mathbb{R}^3$ by

$$T_v(x) := v, \quad R_v(x) := v \times x, \quad x \in \mathbb{R}^3.$$

(a) Compute $[T_v, T_w]$, $[T_v, R_w]$, and $[R_v, R_w]$ for $v, w \in \mathbb{R}^3$.

(b) Write $R_i := R_{\frac{\partial}{\partial x^i}}$. Compute $[R_i, R_j]$.

Solution:

(a) In the lecture we derived the following formula for the Lie bracket of two vector fields X, Y in coordinates $(x^1, \ldots, x^n)$:

$$[X, Y]^j = \sum_{i=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right).$$

As T_v is constant in x we get $[T_v, T_w] = 0$ for all $v, w \in \mathbb{R}^3$. Let us compute the derivative of the function

$$R_v(x) = v \times x = \begin{pmatrix} v^2x^3 - v^3x^2 \\ v^3x^1 - v^1x^3 \\ v^1x^2 - v^2x^1 \end{pmatrix} = \begin{pmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

in standard coordinates on $\mathbb{R}^3$:

$$D_x R_v = \begin{pmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{pmatrix}.$$

So

$$[T_v, R_w]^1 = -v^2w^3 + v^3w^2$$
$$[T_v, R_w]^2 = v^1w^3 - v^3w^1$$
$$[T_v, R_w]^3 = -v^1w^2 + v^2w^1.$$

Hence $[T_v, R_w] = v \times w$. Also

$$
\begin{aligned}
[R_v, R_w]^1 &= -(v^3x^1 - v^1x^3)w^3 + (v^1x^2 - v^2x^1)w^2 + (w^3x^1 - w^1x^3)v^3 - (w^1x^2 - w^2x^1)v^2, \\
&= -((v \times x) \times w)^1 + ((w \times x) \times v)^1 \\
[R_v, R_w]^2 &= (v^2x^3 - v^3x^2)w^3 - (v^1x^2 - v^2x^1)w^1 - (w^2x^3 - w^3x^2)v^3 + (w^1x^2 - w^2w^1)v^1, \\
&= -((v \times x) \times w)^2 + ((w \times x) \times v)^2 \\
[R_v, R_w]^3 &= -(v^2x^3 - v^3x^2)w^2 + (v^3x^1 - v^1x^3)w^1 + (w^2x^3 - w^3x^2)v^2 - (w^3x^1 - w^1x^3)v^1 \\
&= -((v \times x) \times w)^3 + ((w \times x) \times v)^3
\end{aligned}
$$

Hence

$$
\begin{aligned}
[R_v, R_w](x) &= -((v \times x) \times w) - ((x \times w) \times v) \\
&= (w \times v) \times x = R_{w \times v}(x)
\end{aligned}
$$

where we used the Jacobi identity for the cross product.

(b) As $\frac{\partial}{\partial x^1} = (1, 0, 0)$, $\frac{\partial}{\partial x^2} = (0, 1, 0)$, $\frac{\partial}{\partial x^3} = (0, 0, 1)$ and using (a) we get

$$[R_1, R_2](x) = ((0, 1, 0) \times (1, 0, 0)) \times x = (-1, 0, 0) \times x = -R_3(x).$$

Similarly, $[R_1, R_3] = R_2$ and $[R_2, R_3] = -R_1$ and $[R_i, R_i] = 0$.

2. Effect of product on Lie bracket

Let X, Y be differentiable vector fields and f, g differentiable functions on a manifold M. Prove that

$$[fX, gY] = fg[X, Y] + f(X \cdot g)Y - g(Y \cdot f)X.$$

Solution:

In a coordinate chart $(x^1, \ldots, x^n)$ we can locally write

$$[X, Y] = \sum_{j=1}^{n} \sum_{i=1}^{n} \left(X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i} \right) \frac{\partial}{\partial x^j}.$$

Hence we get the formula using the Leibniz rule

$$
\begin{aligned}
[fX, gY] &= \sum_{j=1}^{n}\sum_{i=1}^{n}\left(fX^i\frac{\partial(gY^j)}{\partial x^i} - gY^i\frac{\partial(fX^j)}{\partial x^i}\right)\frac{\partial}{\partial x^j} \\
&= \sum_{j=1}^{n}\sum_{i=1}^{n}\left(fgX^i\frac{\partial Y^j}{\partial x^i} + fX^iY^j\frac{\partial g}{\partial x^i} - fgY^i\frac{\partial X^j}{\partial x^i} - gX^jY^i\frac{\partial f}{\partial x^i}\right)\frac{\partial}{\partial x^j} \\
&= fg\sum_{j=1}^{n}\sum_{i=1}^{n}\left(X^i\frac{\partial Y^j}{\partial x^i} - Y^i\frac{\partial X^j}{\partial x^i}\right)\frac{\partial}{\partial x^j} \\
&\qquad + f\sum_{j=1}^{n}Y^j\sum_{i=1}^{n}X^i\frac{\partial g}{\partial x^i}\frac{\partial}{\partial x^j} - g\sum_{j=1}^{n}X^j\sum_{i=1}^{n}Y^i\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j} \\
&= fg[X,Y] + f(X\cdot g)Y - g(Y\cdot f)X.
\end{aligned}
$$

3. Regularity of solutions

Let $X \in C^k(TM)$ and let $\gamma : (-T, T) \to M$ be a C^1 integral curve of X. Show that γ is C^{k+1}.

Solution:

The order of differentiability is a local property, hence it is enough to consider a vector field and a curve in $\mathbb{R}^n$. An integral curve γ for a vector field X satisfies

$$\frac{\partial\gamma}{dt} = X \circ \gamma.$$

We will use this equation repeatedly. If γ and X are both C^1 also the left hand side $X \circ \gamma$ is C^1. By the equality also the right hand side $\frac{\partial\gamma}{dt}$ is C^1. But by the fundamental theorem of calculus, this means that actually γ is C^2.

Iteratively, if γ and X are C^l then γ is C^{l+1}. So if X is C^k we can get up to γ being C^{k+1}.

4. Closed sets can be obtained as the zero set of a smooth function and can be approximated from outside by regular open sets

(a) Show any closed set $A \subset \mathbb{R}^n$ is the zero set of some smooth function

$$f : \mathbb{R}^n \to \mathbb{R}.$$

(b) Let $A \subset \mathbb{R}^n$ be closed. Show there exist open sets $U_1 \supset U_2 \supset U_3 \supset \dots$ such that ∂U_j is a smooth $(n-1)$-manifold and

$$A = \bigcap_{j=1}^{\infty} U_j.$$

Solution:

(a) Any open set $U \subset \mathbb{R}^n$ can be written as the union of countably many open balls in the following way: Take all balls in U that have rational midpoint and rational radius and are contained in U. As we only have countably many midpoints and countably many radii we get a countable set of balls.

So any closed set $A \subset \mathbb{R}^n$ can be written as the complement countably many open balls in $\mathbb{R}^n$. So write $A = \mathbb{R}^n \setminus \bigcup_{j=1}^{\infty} B_{r_j}(x_j)$ for open balls $B_{r_j}(x_j) \subset \mathbb{R}^n \setminus A$ with rational midpoints and rational radii. Let $\phi_j \in C^\infty(\mathbb{R}^n)$ be bump functions supported in $B_{r_j}(x_j)$ which are equal to 1 on $B_{r_j/2}(x_j)$.

Define

$$f(x) = \sum_{j=1}^{\infty} c_j \phi_j(x)$$

for some real numbers $c_j > 0$ to be chosen after to make sure that the function is well-defined and smooth. By definition $f(x) = 0$ iff $x \in A$. Let the $c_j > 0$ be small enough that

$$c_j ||\phi_j^{(k)}||_\infty < 2^{-j}$$

for all $0 \leq k \leq j$ where $||g||_\infty = \sup_{x \in \mathbb{R}^n} |g(x)|$ of a function $g : \mathbb{R}^n \to \mathbb{R}$ denotes the sup-norm.

With these bounds on the c_j the sum that defines $f(x)$ converges absolutely as well as all derivatives:

$$\begin{aligned} \sum_{j=1}^{\infty} \left| c_j \phi_j^{(k)}(x) \right| &= \sum_{j=1}^{k} \left| c_j \phi_j^{(k)}(x) \right| + \sum_{j=k+1}^{\infty} \left| c_j \phi_j^{(k)}(x) \right| \\ &= \sum_{j=1}^{k} c_j ||\phi_j^{(k)}|| + \sum_{j=k+1}^{\infty} 2^{-j} < \infty \end{aligned}$$

for every $k = 0, 1, \ldots$. Hence the formula for f defines a smooth function with zero set equal to A.

(b) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a non-negative smooth function with zero set A as constructed in part (a). Let $C \subset \mathbb{R}^n$ be the set of critical points of f. By Sard's theorem we have that the Lebesgue measure of the critical values $f(X) \subset \mathbb{R}$ is zero. In particular, there is a monotone decreasing sequence $(x_j)_{j \in \mathbb{N}}$ in $\mathbb{R} \setminus f(X)$ with $x_j > 0$ and $\lim x_j = 0$. Define

$$U_j = f^{-1}((-\infty, x_j)).$$

These sets are open in $\mathbb{R}^n$ because f is continuous. Because the numbers x_j are monotone decreasing sequence converging to 0, the sets U_j satisfy by construction

$$U_1 \supset U_2 \supset U_3 \supset \cdots \supset A$$

with $A = \bigcap_{j=1}^{\infty} U_j$. Moreover, as the x_j are regular points of f the preimages $f^{-1}(\{x_j\})$ are $(n-1)$-dimensional submanifolds of $\mathbb{R}^n$. Note that $f^{-1}(\{x_j\})$ is exactly the boundary of the n-manifold U_j.

5. Covering groups

(a) Let G be a Lie group and K a discrete normal subgroup. The group homomorphism

$$G \to G/K$$

is a covering map. We call it a *covering homomorphism* and G a *covering group* of G/K. For an example see exercise 1 sheet 8.

Hint: Find an open neighborhood of the identity $e \in U$ such that $U \cdot U^{-1} \cap K = \{e\}$.

(b) A discrete normal subgroup K of a connected Lie group G lies in the center of G.

Hint: For $k \in K$ consider the map $g \mapsto gkg^{-1}$.

(c) Find a covering homomorphism

$$S^3 \times S^3 \to SO(4)$$

of degree 2. Since $S^3 \times S^3$ is simply-connected this shows that the universal covering group of $SO(4)$ is $S^3 \times S^3$.

Solution:

(a) To prove that the quotient map $\pi : G \to G/K$ is a covering map, we need to find for each $g \in G$ an open set U_g such that $\{kU_g\}_{k\in K}$ is a disjoint family of open sets. Because then $\pi|_{U_g} : U_g \to K \cdot U_g \subset G/K$ is a homeomorphism and can be used to turn G/K into a manifold. Equipped with this structure $\pi|_{U_g}$ is a diffeomorphism. Moreover, $\pi^{-1}(K \cdot U_g) \cong K \times U_g$, so π is a covering map with fiber K.

So let us prove that for every $g \in G$ there is an open set U_g such that $\{kU_g\}_{k\in K}$ is a disjoint family of open sets. Since $K \subset G$ is discrete there is a neighborhood W of the identity $e \in G$ such that $K \cap W = \{e\}$. By the continuity of multiplication, there is an open neighborhood $V \subset G$ of the identity such that $V \times V \subset W$. Set $U := V \cap V^{-1}$. This is an open set as an intersection of two open sets, it contains the identity and $U \cdot U \subset W$. Moreover, $U^{-1} = U$ because if $u \in U$ then u is in V and $u^{-1} \in V$ hence also $u^{-1} \in U$. So in particular, $U \times U^{-1} \subset W$. Hence $U \cdot U^{-1} \cap K = \{e\}$.

For $g \in G$ define $U_g := gU$. Then the sets $\{kU_g\}_{k \in K}$ are pairwise disjoint: Suppose that there are $k, k' \in K$ such that $k'U_g \cap kU_g = k'gU \cap kgU \neq \emptyset$. Then there are $u, u' \in U$ such that $kgu = k'gu'$. Then

$$u'u^{-1} = g^{-1}k^{-1}k'g \in U \cdot U^{-1} \cap K = \{e\}$$

as the subgroup K is normal. Hence $u' = u$ and $k' = k$. This proves that the sets $\{kU_g\}_{k \in K}$ are pairwise disjoint.

(b) The center of a group G is defined as the elements that commute with all other elements: $Z(G) := \{g \in G \,|\, gh = hg \text{ for all } h \in G\}$.

Suppose K is a discrete normal subgroup of a connected Lie group G. Let $k \in K$. We would like to show that $k \in Z(G)$. Define a map $c_g : G \to G$ by $g \mapsto gkg^{-1}$. The map is continuous and as G is connected, so is the image. On the other hand, the image is also contained in K as K is normal. But a connected subgroup of a discrete group contains only one element. As k is in the image of c_g (for $g = e$, the identity), the image is $c_g(G) = \{k\}$. Hence $gkg^{-1} = k$ for all $g \in G$ which proves that k is in the center of G.

(c) Let S^3 be the group of unit quaternions. Then $S^3 \times S^3$ acts on $\mathbb{R}^4$ by

$$(u, v) \cdot x = uxv^{-1}.$$

This action defines a linear map on $A_{(u,v)} : \mathbb{R}^4 \to \mathbb{R}^4$ for each pair $(u, v) \in S^3 \times S^3$. Moreover, as u, v are of norm 1, the map $A_{(u,v)}$ preserves the norm. Hence the map $A : S^3 \times S^3 \to O(4)$ sending $(u, v) \mapsto A_{(u,v)}$ is a well-defined smooth group homomorphism. The kernel of the map is $\{\pm(1, 1)\} \subset S^3 \times S^3$. So the image is also a manifold of dimension 6 like $S^3 \times S^3$. Because S^3 is connected and $A_{(e,e)} = id \in SO(4)$ the image is actually in $SO(4) \subset O(4)$. The image of A would be a compact submanifold of the connected compact manifold $SO(4)$. But the image of A and $SO(4)$ are both of dimension 6 so the image of A is equal to $SO(4)$. The last argument is proved the same way as exercise 2 sheet 10. An embedding of a compact manifold is a closed and an open map, hence has a connected image.

This proves that $S^3 \times S^3 \to SO(4)$ is a two-sheeted covering map. $S^3 \times S^3$ is the universal cover of $SO(4)$ because $S^3 \times S^3$ is simply-connected.