Exercise Sheet 12

To be handed in until December 13

1. The tangent space of an implicitly defined submanifold

Let $f : M \to N$ be smooth and q a regular value of f. Set $P := f^{-1}(q)$. Prove:

$$T_pP = \ker(D_pf)$$

for all $p \in P$.

Solution:

Let $v \in T_pP$. Suppose that γ is a curve in P with $\gamma(0) = p$ and $\gamma'(0) = v$. Then

$$D_pf(v) = D_p(\gamma'(0)) = D_0(f \circ \gamma) = (f \circ \gamma)'(0) = 0$$

as f restricted to P is constant and γ lies in P, hence $f \circ \gamma$ is constant. This proves

$$T_pP \subset \ker(D_pf).$$

This is an equality for dimension reasons because

$$\dim T_pP = \dim P = \dim M - \dim N = \dim M - \dim \mathrm{im}(D_pf) = \dim(\ker(D_pf)).$$

2. Groups covered by $SU(n)$

Let $n \geq 1$.

(a) Find the center of $SU(n)$.

(b) Show that $SU(n)$ covers only a finite number of groups.
(N.B. $SU(n)$ is simply-connected.)

Solution:

(a) Let $A \in Z(G)$ where $Z(G)$ denotes the center of $SU(n)$. Unitary matrices are diagonalizable with $A = SDS^{-1}$ for a unitary matrix $S \in U(n)$ and a diagonal matrix D. We can assume $S \in SU(n)$ by switching two column vectors in S. Because A is in the center we have $SD = AS = SA$ and because S is invertible $A = D$ is diagonal. For $i \neq j$ let E_{ij} be the matrix that exchanges two column vectors and changes the sign of the columns (corresponds to the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $n = 2$). Then $E_{ij} \in SU(n)$. Again using that A is in the center, $E_{ij}A = AE_{ij}$ implies that the i-th diagonal entry of A is equal to the j-th diagonal entry of A. Hence $A = \lambda I$ for $\lambda \in \mathbb{C}$ and I the identity matrix. Finally, $\det A = 1$ proves that $\lambda^n = 1$, so $\lambda = e^{j2\pi i/n}$ for some $j \in \mathbb{Z}$. The map $\mathbb{Z}_n \to Z(G)$ defined by $j \in \mathbb{Z}_n \to e^{j2\pi i/n}$ is an isomorphism, where $\mathbb{Z}_n$ is the cyclic group of n elements.

(b) If a group homomorphism $\pi : SU(n) \to G$ is a covering it has discrete kernel because it is a covering map and the kernel is normal as the quotient $G \cong SU(n)/\ker\pi$ is a group. We proved in exercise 5b sheet 11 that discrete normal subgroups are subgroups of the center. By part (a) the center is the finite group $\mathbb{Z}_n$, which has only finitely many subgroups $K \subset \mathbb{Z}_n$. So there are only finitely many possibilities $G \cong SU(n)/K$.

3. Relation of flows and Lie brackets

We shall see later that ϕ_X^t commutes with ϕ_Y^t iff $[X, Y] = 0$.

(a) Given $v, w \in \mathbb{R}^3$, recall the two vector fields defined in exercise sheet 11 problem 1:

$$T_v(x) = v, \qquad R_w(x) = w \times x \qquad \text{for } x \in \mathbb{R}^3.$$

Describe the flows $\phi_{T_v}^t$, $\phi_{R_v}^t$ geometrically.

(b) Determine by geometric reasoning conditions on v, w such that the flows $\phi_{T_v}^t$, $\phi_{R_w}^t$ commute.

(c) Determine by computation conditions on v, w such that the Lie brackets $[T_v, R_w]$ vanishes.

Solution:

(a) The flow of T_v is $\phi_{T_v}^t(x) = x + tv$ as

$$\frac{d}{dt}\Big|_{t=0} \phi_{T_v}^t(x) = \frac{d}{dt}\Big|_{t=0}(x + tv) = v.$$

We claim that the flow $\phi_{R_w}^t$ is a rotation with axis w and angle $t|w| \in \mathbb{R}$. The steps are:

(i) $\phi_{R_w}^t$ fixes points on the line $\mathbb{R}w$.

(ii) The distance of $\phi^t_{R_w}(x)$ to the line $\mathbb{R}w$ stays constant when we vary t and fix x.

(iii) $|\phi^t_{R_w}(x)|$ stays constant.

(iv) $\phi^t_{R_w}(x)$ rotates with constant speed $|w|$ around w when varying t.

Indeed:

(i) If $x \in \mathbb{R}w$ then $R_w(x) = 0$ hence $\phi^t_{R_w}(x) = x$, so the flow fixes the axis $\mathbb{R}w$.

(ii) Let $x \in \mathbb{R}^3$ be arbitrary. The distance of the point $x(t) := \phi^t_{R_w}(x)$ to axis $\mathbb{R}w$ is given by the formula

$$\begin{aligned} d(x(t), \mathbb{R}w) &= \left| x(t) - \frac{\langle x(t), w\rangle w}{|w|^2} \right| \\ &= \frac{|\langle w, w\rangle x(t) - \langle x(t), w\rangle w|}{|w|^2} \\ &= \frac{|w \times (x(t) \times w)|}{|w|^2} \\ &= \frac{|x(t) \times w|}{|w|} \end{aligned}$$

This distance is constant in t as

$$\begin{aligned} \frac{d}{dt} d(x(t), \mathbb{R}w)^2 &= \frac{d}{dt} \frac{|x(t) \times w|^2}{|w|^2} \\ &= \frac{\langle x'(t) \times w, x(t) \times w\rangle}{|w|^2} \\ &= \frac{\langle (w \times x(t)) \times w, x(t) \times w\rangle}{|w|^2} \\ &= 0 \end{aligned}$$

where we used that $(w \times x(t)) \times w$ is orthogonal to $x(t) \times w$.

(iii) This follows from

$$\frac{d}{dt}\langle x(t), x(t)\rangle = 2\langle x'(t), x(t)\rangle = 2\langle w \times x(t), x(t)\rangle = 0.$$

(iv) Combining (ii) and (iii) we conclude that $x(t) = \phi^t_{R_w}(x)$ stays on the circle around $\mathbb{R}w$ going through x and with the smallest distance from $\mathbb{R}w$. We are left to show that $x(t)$ rotates in constant speed $|w|$ around $\mathbb{R}w$. The speed of rotation is $|R_w(x)| = |w \times x|$ and happens at a distance $d(x(t), \mathbb{R}w) = \frac{|x(t)\times w|}{|w|} = \frac{|x \times w|}{|w|}$ from the axis. So the angular speed is $|w|$. The rotation direction follows the right-hand rule for given directed axis w.

(b) The flows commute iff one of the vectors v and w is zero or if they are parallel.

(c) In sheet 11 exercise 1 we already computed $[T_v, R_w] = v \times w$ which is zero whenever one of the vectors v and w is zero or if they are parallel.

4. Vector fields on matrix manifold

Fix a matrix $A \in \mathbb{R}^{n\times n}$. Define a vector field on $\mathbb{R}^{n\times n}$ by

$$X_A(B) = AB, \qquad \text{for } B \in \mathbb{R}^{n\times n}.$$

(a) Show that X_A is complete.

(b) Find a formula for the flow $\phi^t_{X_A}$.

Solution:

(a) The vector field has linear growth as

$$|X_A(B)| = |AB| \leq |A||B|$$

where the norm of a matrix is $|C| = \sqrt{\sum_{i=1,j=1}^n C_{ij}^2}$. Vector fields with linear growth are complete by a theorem from the lecture.

(b) The matrix exponential $\phi^t_{X_A}(B) = e^{tA}B$ satisfies $\phi^0_{X_A}(B) = B$ and

$$\frac{d}{dt}\phi^t_{X_A}(B) = Ae^{tA}B = X_A(\phi^t_{X_A}(B)).$$

5. An explicit example of a flow

Let X be the vector field on $\mathbb{R}^2$ given by $X(x,y) = (rx - x - y, ry + x - y)$ where $r = \sqrt{x^2+y^2}$.

(a) Draw the vector field X.

(b) Find the flow ϕ^t_X of X.

Hint: Work in polar coordinates.

(c) Draw the (open) domain of definition of ϕ_X.

(d) Is X complete? Restricted to which open subsets of $\mathbb{R}^2$ is the vector field X complete?

Solution:

(a) In polar coordinates (r, θ), the vector field is $X(r, \theta) = (r(r-1), 1)$. The constant θ component tells me that there is a constant circular component of the vector field such that the flows will do circles around the origin. On the circle $r = 1$ the radial component is 0, so the vector field is the unit tangent to the circle and the flow simply goes around the circle. For $r > 1$ the vector field points in radial direction outwards (as $r(r-1)$ is positive) plus the constant circular direction. So the flow starting at a point with $r > 1$ will move away from 0 in spirals. For $r < 1$ the vector field points inside plus the constant circular direction. So the flow starting at a point with $r < 1$ will move towards 0 in spirals.

(b) The angular coordinate of the flow is $\theta(t) = \theta + t$. For the radial coordinate we need to solve the equation $r' = r(r-1)$. By separating the variables

$$\int \frac{dr}{r(r-1)} = \int dt$$

we get

$$\int \frac{dr}{r(r-1)} = \int \frac{dr}{r-1} - \int \frac{dr}{r} = \ln|r-1| - \ln|r| = t + C.$$

For $r \geq 1$, exponentiating yields $\frac{r-1}{r} = e^{t+C}$ and solving for r gives $r(t) = \frac{1}{1-e^{t+C}}$. With $r(0) = r_0$ we can find the constant C and get

$$r(t) = \frac{1}{1 - e^t(1 - 1/r_0)}.$$

Hence the flow of X is

$$\phi_X^t(r, \theta) = \left(\frac{1}{1 - e^t(1 - 1/r)}, \theta + t \right)$$

for $r \geq 1$ and similarly, the same equation holds for $0 \leq r \leq 1$. The flow is not defined at time t such that $1 - e^t(1 - 1/r) = 0$. This is only possible for $r > 1$ and happens at $t = \ln\left(\frac{1}{1-1/r}\right)$.

(c) The domain of definition of ϕ_X is

$$\left\{ (t, r, \theta) \,\middle|\, -\infty < t < \ln\left(\frac{1}{1 - 1/r}\right) \text{ if } r > 1 \text{ and } -\infty < t < \infty \text{ if } r \leq 1 \right\}.$$

(d) The largest open set to which X can be restricted and is complete is the open unit ball as a point outside the unit ball escapes in finite positive time to infinity. For points in the unit ball, the flow is defined for all time. If an open complete set contains 0 then it is the unit ball as the negative flow expands the open set U to all of the unit ball. Moreover, we can get any open set where X is complete by the following: Let $V \subset \frac{1}{2}S^1 = \{x \in \mathbb{R}^2 \,|\, |x| = 1/2\}$ be open. Then

$$U_V := \{\phi_X^t(x) \,|\, t \in \mathbb{R}, x \in V\}$$

is open in $\mathbb{R}^2$ and note that X restricted to U_V is complete.