Exercise Sheet 13

To be handed in until December 20

1. Pushforward and pullback of vector fields

Let $\phi : M \to N$ be smooth, X a smooth vector field on M and Y a smooth vector field on N.

Define the *pushforward of X by ϕ* via

$$\phi_*(X)(q) := D_{\phi^{-1}(q)}\phi(X(\phi^{-1}(q))).$$

Define the *pullback of Y by ϕ* via

$$\phi^*(Y)(p) := (D_p\phi)^{-1}(Y(\phi(p))).$$

(a) Show that if ϕ is bijective, $\phi_*(X)$ is defined.

(b) Show that if ϕ is a diffeomorphism, $\phi_*(X) \in C^\infty(TN)$.

(c) Give an example where ϕ is bijective but $\phi_*(X)$ not smooth.

(d) Show that if ϕ is a local diffeomorphism, $\phi^*(Y)$ is defined and is in $C^\infty(TM)$.

(e) Suppose $\phi : M \to N$ and $\psi : N \to P$ are diffeomorphisms. Show

$$\phi^*\psi^* = (\psi \circ \phi)^*, \quad \psi_*\phi_* = (\psi \circ \phi)_*,$$

$$\phi^*\phi_* = id_{C^\infty(TM)}, \quad (\phi^{-1})^* = \phi_*.$$

Solution:

(a) By definition.

(b) $\phi_*(X)$ is the composition of the smooth maps

$$N \xrightarrow{\phi^{-1}} M \xrightarrow{X} TM \xrightarrow{D\phi} TN.$$

(c) Let $\phi : \mathbb{R} \to \mathbb{R}$ be given by $x \mapsto x^3$. This map is bijective but its inverse $\phi^{-1} : \mathbb{R} \to \mathbb{R}$ given by

$$\phi^{-1}(x) = \begin{cases} \sqrt[3]{x}, & x \geq 0 \\ -\sqrt[3]{-x}, & x < 0 \end{cases}$$

is not smooth at 0. The pushforward of the vector field $X(y) = 1$ is

$$\phi_*(X)(x) = D_{\phi^{-1}(x)}\phi(X(\phi^{-1}(x))) = \phi'(\phi^{-1}(x)) = \begin{cases} 3\sqrt[3]{x^2}, & x \geq 0, \\ 3\sqrt[3]{-x^2}, & x < 0, \end{cases}$$

which is not smooth at $x = 0$.

(d) The definition is well-defined because $d\phi_p$ is invertible if ϕ is a local diffeomorphism. Moreover, the pushforward is smooth if Y is smooth as the composition of the smooth maps

$$M \xrightarrow{\phi} N \xrightarrow{Y} TN \xrightarrow{D\phi^{-1}} TM.$$

(e) Let Z be a vector field on P and $p \in M$. Then the double pullback is

$$\begin{aligned} \phi^*\psi^*(Z)(p) &= (D_p\phi)^{-1}\psi^*(Z)(\phi(p)) \\ &= (D_p\phi)^{-1}(D_{\phi(p)}\psi)^{-1}(Z(\psi \circ \phi(p))) \\ &= (D_{\phi(p)}\psi D_p\phi)^{-1}(Z(\psi \circ \phi(p))) \\ &= (D_p(\psi \circ \phi))^{-1}(Z(\psi \circ \phi(p))) \\ &= (\psi \circ \phi)^*(Z)(p). \end{aligned}$$

Let X be a vector field on M and $r \in P$. The double pushforward is

$$\begin{aligned} \psi_*\phi_*(X)(r) &= D_{\psi^{-1}(r)}\psi(\phi_*(X)(\psi^{-1}(r))) \\ &= (D_{\psi^{-1}(r)}\psi D_{\phi^{-1}(\psi^{-1}(r))})\phi(X(\phi^{-1}\psi^{-1}(r))) \\ &= D_{(\psi\circ\phi)^{-1}(r)}(\psi \circ \phi)(X((\psi \circ \phi)^{-1}(r))) \\ &= (\psi \circ \phi)_*(X)(r). \end{aligned}$$

Let X be a vector field on M and $p \in M$. Then

$$\begin{aligned} \phi^*\phi_*(X)(p) &= (D_p\phi)^{-1}(\phi_*(X)(\phi(p))) \\ &= (D_p\phi)^{-1}D_{\phi^{-1}(\phi(p))}\phi(X(\phi^{-1}(\phi(p)))) \\ &= X(p). \end{aligned}$$

The last property follows from the previous ones

$$(\phi^{-1})^* = (\phi^{-1})^*(\phi^*\phi_*) = ((\phi^{-1})^*\phi^*)\phi_* = (\phi \circ \phi^{-1})^*\phi_* = id^*\phi_* = \phi_*.$$

2. Another flow example

Let $M = \mathbb{R}^2 \setminus \{0\}$ and consider the vector field $X(p) = \frac{\partial}{\partial x}$ for all $p \in M$. Define $b(p)$ to be the maximal (positive) time of existence for the flow of X starting at $p \in M$.

(a) Compute $b(p)$ for all $p \in M$.

(b) Verify that
$$b(p) = \liminf_{q \to p} b(q)$$
for all $p \in M$.

(c) Find the points where $\lim_{q \to p} b(q)$ does not exist.

(d) Verify directly that the maximal domain of existence
$$\mathcal{U} = \{(p,t) \,|\, x \in M, a(p) < t < b(p)\}$$
is open.

Solution:

The flow is $\phi_X^t((x,y)) = (x+t, y)$.

(a) Only points on $I = \{(x,0) \,|\, x < 0\} \subset M$ cannot be flowed for infinite positive time. We have
$$b(p) = \begin{cases} \infty, & p \notin I, \\ -x, & p \in I. \end{cases}$$

(b) If $p \notin I$ then $\lim_{q \to p} b(q)$ as the set $\mathbb{R}\backslash I$ is open and b is infinity on this set. On I the function b is not continuous as there are arbitrarily close points with where b is ∞. Still $b(p) = \liminf_{q \to p} b(q)$ holds.

(c) $\mathcal{U}$ is the union of open sets and thus open:
$$\mathcal{U} = \Big(M \setminus \mathbb{R} \times \{0\}) \times \mathbb{R}\Big) \cup \{(x,y,t) \,|\, x > 0, y \in \mathbb{R}, -x < t < \infty\}$$
$$\cup \{(x,y,t) \,|\, x < 0, y \in \mathbb{R}, -\infty < t < -x\}$$

3. Left-invariant and right-invariant vector fields on matrix groups

Let $G = Gl(n, \mathbb{R}) \subset \mathbb{R}^{n \times n}$ be the (Lie) group of invertible matrices. For A in $\mathbb{R}^{n \times n}$ define vector fields $X_A, Y_A \in C^\infty(TG)$ by
$$X_A(B) := AB, \qquad Y_A(B) := BA$$
for $B \in G$.

For $C \in G$ define maps $L_C, R_C : G \to G$ by
$$L_C(B) := CB, \qquad R_C(B) := BC^{-1}$$
for $B \in G$. These maps are called *left and right translation by C*.

(a) Verify

$$L_C \circ L_D = L_{CD}, \qquad R_C \circ R_D = R_{CD}.$$

(b) Conclude that L, R are injective homomorphisms $L, R : G \to \mathrm{Diff}(G)$.

(c) We call a vector field $Z \in C^\infty(TG)$

- *left-invariant* if $L_C^*(Z) = Z$ for all $C \in G$,
- *right-invariant* if $R_C^*(Z) = Z$ for all $C \in G$.

Which of X_A, Y_A is left/right-invariant?

(d) Show that any left-invariant or right-invariant vector field on G is either of the form X_A or Y_A.

Solution:

(a)

$$L_C \circ L_D(B) = L_C(DB) = CDB = L_{CD}(B)$$
$$R_C \circ R_D(B) = R_C(BD^{-1}) = BD^{-1}C^{-1} = B(CD)^{-1} = R_{CD}(B)$$

(b) Direct consequence

(c) Y_A is left-invariant:

$$\begin{aligned} L_C^*(Y_A)(B) &= (D_B L_C)^{-1}(Y_A(L_C(B))) \\ &= C^{-1}(Y_A(CB)) \\ &= C^{-1}CBA \\ &= BA \\ &= Y_A(B). \end{aligned}$$

X_A is left-invariant:

$$\begin{aligned} R_C^*(X_A)(B) &= (D_B R_C)^{-1}(X_A(R_C(B))) \\ &= (X_A(BC^{-1}))C \\ &= ABC^{-1}C \\ &= AB \\ &= X_A(B). \end{aligned}$$

(d) Denote $I \in G$ the identity matrix and let Z be a vector field with $X(I) = A$. If Z is left-invariant then left-invariance for $C = B^{-1}$ ensures

$$\begin{aligned} Z(B) &= L_{B^{-1}}^*(Z)(B) \\ &= (D_B L_{B^{-1}})^{-1}(Z(L_{B^{-1}}(B))) \\ &= BZ(I) = BA = Y_A(B). \end{aligned}$$

Similarly, if Z is right-invariant then right-invariance for $C = B$ ensures

$$\begin{aligned} Z(B) &= R_B^*(Z)(B) \\ &= (D_B R_B)^{-1}(Z(R_B(B))) \\ &= Z(I)B = AB = X_A(B). \end{aligned}$$

4. The Lie bracket and the matrix commutator

In this exercise we will show that

$$[Y_A, Y_B] = Y_{[A,B]}$$

where $[Y_A, Y_B]$ is calculated as the Lie bracket of vector fields and $[A, B]$ is the matrix commutator in $\mathbb{R}^{n\times n}$. In other words, the map $A \mapsto Y_A$, from matrices to vector fields, is a Lie algebra homomorphism.

(a) Show that the flow of Y_A is $\phi^t_{Y_A}(C) = Ce^{tA}$ for $C \in G = Gl(n, \mathbb{R})$.

(b) Show that the derivative of $\phi^t_{Y_A}$ at C is $D_C\phi^t_{Y_A}(E) = Ee^{tA}$ for $E \in \mathbb{R}^{n\times n}$.

(c) We will see next week that the Lie derivative agrees with the Lie bracket. Compute $[Y_A, Y_B]$ using that $[Y_A, Y_B](C) = \mathcal{L}_{Y_A} Y_B(C)$ for all $C \in G$.

Solution:

(a) Indeed, $\phi^0_{Y_A}(C) = C$ and

$$\frac{d}{dt}\phi^t_{Y_A}(C) = Ce^{tA}A = Y_A(\phi^t_{Y_A}(C)).$$

(b)

$$\begin{aligned} D_C\phi^t_{Y_A}(E) &= \frac{d}{ds}\Big|_{s=0}\phi^t_{Y_A}(C + sE) \\ &= \frac{d}{ds}\Big|_{s=0}(C + sE)e^{tA} \\ &= Ee^{tA} \end{aligned}$$

(c)

$$
\begin{aligned}
[Y_A, Y_B](C) &= \mathcal{L}_{Y_A} Y_B(C) \\
&= \frac{d}{dt}\Big|_{t=0} (\phi^t_{Y_A})^*(Y_B)(C) \\
&= \frac{d}{dt}\Big|_{t=0} (D_C \phi^t_{Y_A})^{-1}(Y_B(\phi^t_{Y_A}(C))) \\
&= \frac{d}{dt}\Big|_{t=0} (D_C \phi^t_{Y_A})^{-1}(Y_B(Ce^{tA})) \\
&= \frac{d}{dt}\Big|_{t=0} (D_C \phi^t_{Y_A})^{-1}(Ce^{tA}B)) \\
&= \frac{d}{dt}\Big|_{t=0} Ce^{tA}Be^{-tA} \\
&= C\left(\frac{d}{dt}\Big|_{t=0} e^{tA}\right) B + CB\left(\frac{d}{dt}\Big|_{t=0} e^{-tA}\right) \\
&= C(AB - BA) \\
&= C[A, B] \\
&= Y_{[A,B]}(C)
\end{aligned}
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