Exercise Sheet 14

Not to be handed in

1. Another proof of the Jacobi identity

(a) Let X, Y, Z be smooth vector fields on a manifold. Prove the Jacobi identity

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$$

using the identity

$$\phi^*[Y,Z] = [\phi^* Y, \phi^* Z].$$

for $\phi = \phi^t$, the local flow of X, and then differentiating at $t = 0$.

(b) A derivation on an algebra $(A, \cdot)$ is a linear map $D : A \to A$ that satisfies the Leibniz rule:

$$D(y \cdot z) = (Dy) \cdot z + y \cdot (Dz).$$

Prove that the Lie derivative L_X is a derivation on the nonassociative algebra $(C^\infty(TM), [\cdot,\cdot])$.

Solution:

(a) Taking the derivative of $(\phi^t)^*[Y,Z] = [(\phi^t)^*Y, (\phi^t)^*Z]$ at $t = 0$ on both sides yields

$$\frac{d}{dt}\Big|_{t=0} (\phi^t)^*[Y,Z] = L_X[Y,Z] = [X,[Y,Z]] = -[[Y,Z],X]$$

and

$$\begin{aligned}
\frac{d}{dt}\Big|_{t=0}[(\phi^t)^*Y, (\phi^t)^*Z] &= \left[\frac{d}{dt}\Big|_{t=0}(\phi^t)^*Y, (\phi^0)^*Z\right] + \left[(\phi^0)^*Y, \frac{d}{dt}\Big|_{t=0}(\phi^t)^*Z\right] \\
&= [L_X Y, Z] + [Y, L_X Z] \\
&= [[X,Y],Z] + [Y,[X,Z]] \\
&= [[X,Y],Z] - [Y,[Z,X]] \\
&= [[X,Y],Z] + [[Z,X],Y]
\end{aligned}$$

where we used that $[\cdot,\cdot]$ is bilinear and switching entries produces a minus. So

$$-[[Y,Z],X] = \frac{d}{dt}\Big|_{t=0}(\phi^t)^*[Y,Z] = \frac{d}{dt}\Big|_{t=0}[(\phi^t)^*Y, (\phi^t)^*Z] = [[X,Y],Z] + [[Z,X],Y]$$

proves the Jacobi identity.

(b) This is another formulation of the Jacobi identity:

$$\begin{aligned} L_X([Y,Z]) &= [X,[Y,Z]] \\ &= -[[Y,Z],X] \\ &\overset{\text{Jacobi}}{=} [[X,Y],Z] + [[Z,X],Y] \\ &= [[X,Y],Z] + [Y,[X,Z]] \\ &= [L_X Y, Z] + [Y, L_X Z]. \end{aligned}$$

2. Commutation error

Let ϕ_X^s, ϕ_Y^t be the flows of two vector fields X, Y. Show that the following formulas

(a) $\phi_Y^t \circ \phi_X^s(x) = x + sX + tY + O(|s|^2 + |t|^2)$

(b) $\phi_Y^t \circ \phi_X^s(x) - \phi_X^s \circ \phi_Y^t(x) = st[X,Y] + O(|s|^3 + |t|^3)$

(c) $\phi_Y^{-t} \circ \phi_X^{-s} \circ \phi_Y^t \circ \phi_X^s(x) = x + st[X,Y] + O(|s|^3 + |t|^3)$

hold in any coordinate system.

Solution:

As the statements are all local we may assume that X, Y are vector fields on $\mathbb{R}^n$. Also, fix a point $x \in \mathbb{R}^n$. Define $f : (-\delta, \delta) \times (-\delta, \delta) \to \mathbb{R}^n$ by

$$f(t,s) = \phi_Y^t \circ \phi_X^s(x)$$

where $\delta > 0$ is chosen small enough such that the flows are defined. Taylor expansion for f at $(t,s) = 0$ is

$$\begin{aligned} f(t,s) = f(0,0) &+ \frac{\partial f}{\partial t}(0,0)t + \frac{\partial f}{\partial s}(0,0)s && (1) \\ &+ \frac{\partial^2 f}{\partial t^2}(0,0)\frac{t^2}{2} + \frac{\partial^2 f}{\partial t \partial s}(0,0)st + \frac{\partial^2 f}{\partial s^2}(0,0)\frac{s^2}{2} && (2) \\ &+ O(|s|^3 + |t|^3) && (3) \end{aligned}$$

We have $f(0,0) = x$ and first derivatives:

$$\begin{aligned} &f(0,0) = x \\ &\frac{\partial f}{\partial t}(t_0, s_0) = \frac{d}{dt}\Big|_{t=t_0} f(t, s_0) = \frac{d}{dt}\Big|_{t=t_0} \phi_Y^t(\phi_X^{s_0}(x)) = Y(\phi_Y^{t_0}(\phi_X^{s_0}(x))) \\ &\frac{\partial f}{\partial s}(0, s_0) = \frac{d}{ds}\Big|_{s=s_0} f(0,s) = \frac{d}{dt}\Big|_{s=s_0} \phi_X^s(x) = X(\phi_X^{s_0}(x)) \end{aligned}$$

The second derivatives are

$$\frac{\partial^2 f}{\partial t^2}(0,0) = \frac{d}{dt}\Big|_{t=0}\frac{\partial f}{\partial t}(t,0) = \frac{d}{dt}\Big|_{t=0} Y(\phi_Y^t(x)) = D_x Y \frac{d}{dt}\Big|_{t=0}\phi_Y^t(x) = D_x Y(Y(x)) = D_Y Y(x)$$
$$\frac{\partial^2 f}{\partial s \partial t}(0,0) = \frac{d}{ds}\Big|_{s=0}\frac{\partial f}{\partial t}(0,s) = \frac{d}{ds}\Big|_{s=0} Y(\phi_X^s(x)) = D_x Y \frac{d}{ds}\Big|_{s=0}\phi_X^s(x) = D_x Y(X(x)) = D_X Y(x)$$
$$\frac{\partial^2 f}{\partial s^2}(0,0) = \frac{d}{ds}\Big|_{s=0}\frac{\partial f}{\partial s}(0,s) = \frac{d}{ds}\Big|_{s=0} X(\phi_X^s(x)) = D_x X \frac{d}{ds}\Big|_{s=0}\phi_X^s(x) = D_x X(X(x)) = D_X X(x)$$

(a) Follows from

$$f(0,0) = x, \qquad \frac{\partial f}{\partial t}(0,0) = Y(x), \qquad \frac{\partial f}{\partial s}(0,0) = X(x)$$

(b) Follows by the computations above. All terms cancel when taking the difference except $st(D_X Y(x) - D_Y X(x))$ which is by definition $st[X,Y](x)$.

(c) By (b) we get

$$\phi_Y^{-t} \circ \phi_X^{-s}(y) - \phi_X^{-s} \circ \phi_Y^{-t}(y) = st[X,Y] + O(|s|^3 + |t|^3)$$

Evaluated at $y = \phi_Y^t \circ \phi_X^s(x)$ we get

$$\phi_Y^{-t} \circ \phi_X^{-s} \circ \phi_Y^t \circ \phi_X^s(x) - x = st[X,Y] + O(|s|^3 + |t|^3)$$

which proves (c).

3. Old question, new computation

Given $v, w \in \mathbb{R}^3$, recall the vector fields defined in exercise sheet 11 problem 1:

$$T_v(x) = v, \qquad R_w(x) = w \times x \qquad \text{for } x \in \mathbb{R}^3.$$

We already computed the Lie brackets of these vector fields. Compute

$$L_{T_v} T_w, \quad L_{T_v} R_w, \quad L_{R_w} T_v, \quad L_{R_v} R_w$$

directly using the definition of the Lie derivative.

Solution:

Recall that the flow of T_v is $\phi_{T_v}^t(x) = x + tv$, and that the flow $\phi_{R_w}^t$ is a rotation with axis w and angle $t|w| \in \mathbb{R}$. But as $R_w(x) = w \times x$ is a linear map. The flow of R_w is also

$$\phi_{R_w}^t(x) = e^{tR_w}(x)$$

because

$$\frac{d}{dt} e^{tR_w}(x) = R_w(e^{tR_w}(x)).$$

Since $\phi^t_{R_w}(x) = e^{tR_w}(x)$ is a linear map in x also

$$D_x\phi^t_{R_w}(y) = D_x e^{tR_w}(y) = e^{tR_w}(y)$$

for all $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$. Also

$$D_x\phi^t_{T_v}(y) = y = id_{\mathbb{R}^3}(y)$$

for all $x \in \mathbb{R}^3$ and $y \in \mathbb{R}^3$.

We can now compute the Lie derivatives:

$$\begin{aligned}
L_{T_v}T_w(x) &= \frac{d}{dt}\Big|_{t=0}(\phi^t_{T_v})^* T_w(x) \\
&= \frac{d}{dt}\Big|_{t=0}(D_x\phi^t_{T_v})^{-1}(T_w(\phi^t_{T_v}(x))) \\
&= \frac{d}{dt}\Big|_{t=0}(id_{\mathbb{R}^3}(T_w(x+tv))) \\
&= \frac{d}{dt}\Big|_{t=0}(id_{\mathbb{R}^3}(w)) \\
&= \frac{d}{dt}\Big|_{t=0}(w) = 0
\end{aligned}$$

$$\begin{aligned}
L_{T_v}R_w(x) &= \frac{d}{dt}\Big|_{t=0}(\phi^t_{T_v})^* R_w(x) \\
&= \frac{d}{dt}\Big|_{t=0}(D_x\phi^t_{T_v})^{-1}(R_w(\phi^t_{T_v}(x))) \\
&= \frac{d}{dt}\Big|_{t=0}(id_{\mathbb{R}^3}(R_w(x+tv))) \\
&= \frac{d}{dt}\Big|_{t=0}(id_{\mathbb{R}^3}(w \times (x+tv))) \\
&= \frac{d}{dt}\Big|_{t=0}(w \times (x+tv)) = w \times v
\end{aligned}$$

$$\begin{aligned}
L_{R_w}T_v(x) &= \frac{d}{dt}\Big|_{t=0}(\phi^t_{R_w})^* T_v(x) \\
&= \frac{d}{dt}\Big|_{t=0}(D_x\phi^t_{R_w})^{-1}(T_v(\phi^t_{R_w}(x))) \\
&= \frac{d}{dt}\Big|_{t=0}(e^{-tR_w}(T_v(e^{tR_w}(x)))) \\
&= \frac{d}{dt}\Big|_{t=0}(e^{-tR_w}(v)) \\
&= -R_w(e^{-0R_w}(v)) \\
&= -R_w(v) = -w \times v
\end{aligned}$$

$$
\begin{aligned}
L_{R_v} R_w(x) &= \frac{d}{dt}\Big|_{t=0} (\phi^t_{R_w})^* R_w(x) \\
&= \frac{d}{dt}\Big|_{t=0} (D_x \phi^t_{R_v})^{-1} (R_w(\phi^t_{R_v}(x))) \\
&= \frac{d}{dt}\Big|_{t=0} (e^{-tR_v}(R_w(e^{tR_v}(x)))) \\
&= \frac{d}{dt}\Big|_{t=0} (e^{-tR_v}(w \times e^{tR_v}(x))) \\
&= \left(\frac{d}{dt}\Big|_{t=0} e^{-tR_v}\right)(w \times e^{0R_v}(x)) + \left(e^{-0R_v}\left(w \times \frac{d}{dt}\Big|_{t=0}\left(e^{tR_v}\right)(x)\right)\right) \\
&= -R_v(w \times x) + (w \times R_v(x)) \\
&= -v \times (w \times x) + w \times (v \times x) \\
&= (w \times x) \times v + (x \times v) \times w \\
&= -(v \times w) \times x = -R_{v \times w}(x) = R_{w \times v}(x)
\end{aligned}
$$

where we used the Jacobi identity for the cross product to get to the last line.

4. Twigs on a stream

Let X, Y be smooth vector fields. Consider the vector field

$$Y^t = (\phi^t_X)_*(Y).$$

(a) Consider the vector field

$$X(x, y) = \frac{\partial}{\partial x} - y g(x) \frac{\partial}{\partial y}$$

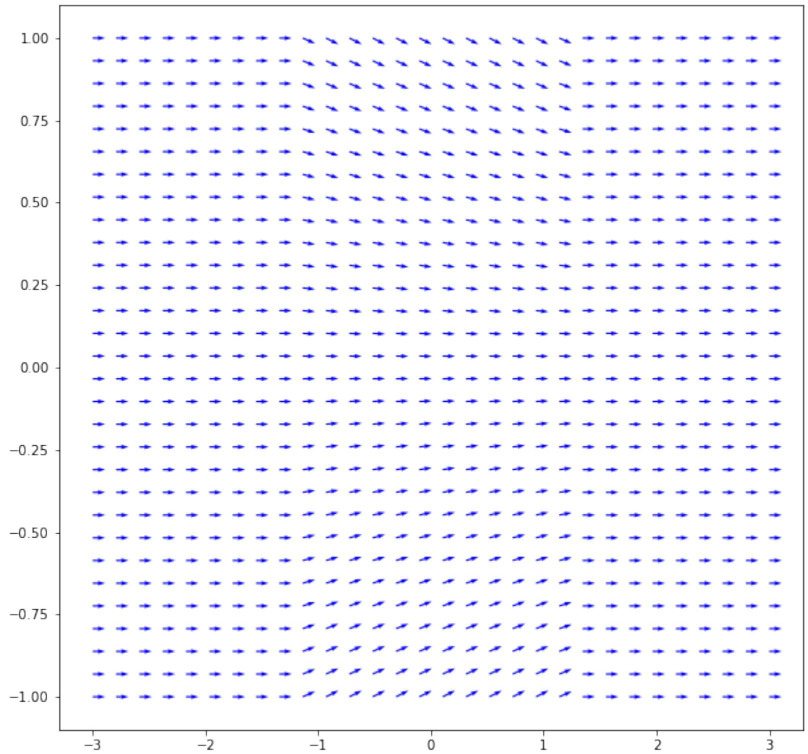
where $g : \mathbb{R} \to [0, 1]$ is a bump function which is 1 on $[-1, 1]$ and 0 outside $[-2, 2]$. Draw the vector field X, and the flow ϕ^t_X by sketching some integral curves.

(b) Draw Y^t for $Y = \frac{\partial}{\partial y}$ for different times e.g. $t = 0, 1, 2, 5$.

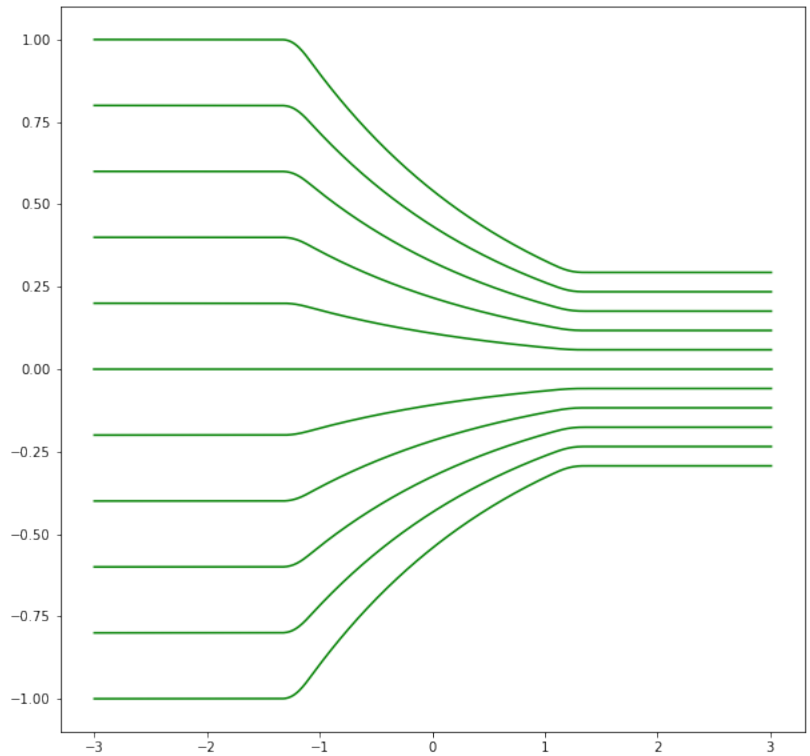
(c) A metaphor for the vector field Y^t is "twigs on a stream". Explain. Is this a good metaphor?

Solution:

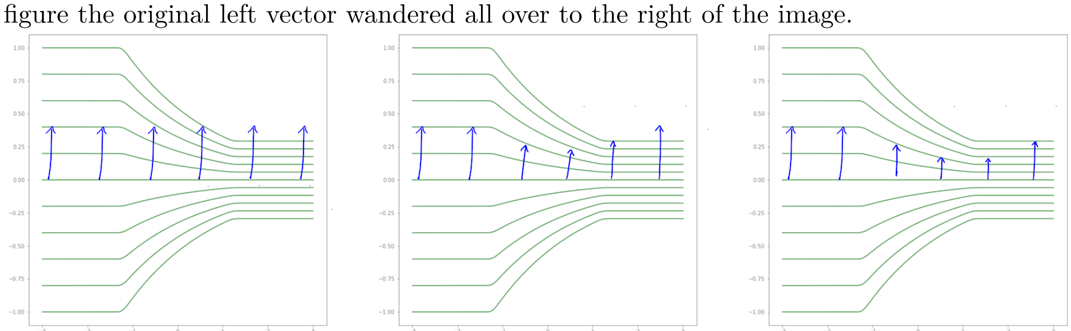
(a) The vector field is

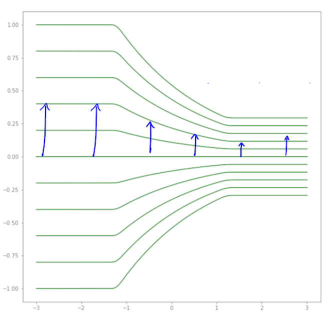


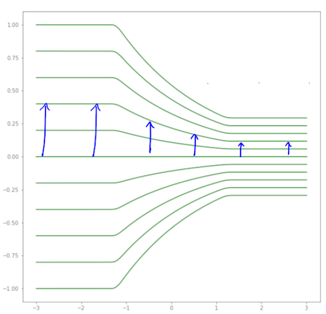
and has integral curves



(b) Let us draw qualitatively some arrows of Y^t for points on the x-axis. If an arrow crosses two integral lines, then the pushed forward vector also crosses two integral lines. The first figure shows the initial vector field. The second figure shows the vector field after moving by $t = 1$, where the time 1 means that a vector was transported one more to the right. So in the fifth figure the original left vector wandered all over to the right of the image.







(c) To get the directions, yes. But pushforwards can also change the *length* of vector fields. Also, water is incompressible, so ϕ_X^t should be volume preserving, i.e. $\det D_p\phi_X^t = 1$ for all p. Not all vector fields satisfy that.

5. Parking is difficult

A car moves in the plane $\mathbb{R}^2$, identified with $\mathbb{C}$. The movement of the car is given by its position $(x(t), y(t)) \in \mathbb{R}^2$ and its direction given by the unit vector $\theta \in S^1$. Moreover, we assume that the direction of movement always coincides with the direction of the car. Now consider the vector fields

$$X(x, y, \theta) := (\cos\theta, \sin\theta, 1)$$
$$Y(x, y, \theta) := (\cos\theta, \sin\theta, -1)$$

on the configuration space $M := \mathbb{R}^2 \times S^1$.

(a) What happens to the car if it moves by X? If it moves by Y?

(b) Compute $[X, Y]$.

(c) Why is parking so difficult? Explain fully and carefully.

Solution:

(a) The vector field X is in $\mathbb{R}^2$-direction the vector $(\cos\theta, \sin\theta)$ which points in direction of the car. In S^1-direction the vector 1 is pointing counterclockwise, i.e. to the left from the point of view of the car direction. The same for Y but the S^1-direction is pointing right. The flows of the vector field are

$$\phi_X^t(x, y, \theta) = (x + \sin(\theta + t) - \sin\theta, y - \cos(\theta + t) + \cos\theta, \theta + t)$$
$$\phi_Y^t(x, y, \theta) = (x - \sin(\theta - t) + \sin\theta, y + \cos(\theta - t) - \cos\theta, \theta - t)$$

because

$$\begin{aligned}\frac{d}{dt}\phi_X^t(x,y,\theta) &= (\cos(\theta+t), \sin(\theta+t), 1) \\ &= X\left(\phi_X^t\left(x,y,\theta\right)\right)\end{aligned}$$

and

$$\phi_X^0(x,y,\theta) = (x,y,\theta)$$

Similarly for Y. So for fixed starting configuration (x,y,θ) the car moves in $\mathbb{R}^2$ with X counterclockwise in a circle of radius 1 and initial direction θ. So the midpoint of the circle is $(x-\sin\theta, y+\cos\theta)$. Similarly for Y: For fixed starting configuration (x,y,θ) the car moves in $\mathbb{R}^2$ with Y clockwise in a circle of radius 1 and initial direction θ. So the midpoint of the circle is $(x+\sin\theta, y-\cos\theta)$.

(b) In coordinates $(x,y,\theta) = (z,u)$ with $z \in \mathbb{C}$ and $u \in S^1$ we have $\widetilde{X}(z,u) = (u, iu)$ and $\widetilde{Y}(z,u) = (u, -iu)$ we compute

$$\begin{aligned}[\widetilde{X},\widetilde{Y}]^1 &= \widetilde{X}^1\frac{\partial\widetilde{Y}^1}{\partial z} - \widetilde{Y}^1\frac{\partial\widetilde{X}^1}{\partial z} + \widetilde{X}^2\frac{\partial\widetilde{Y}^1}{\partial u} - \widetilde{Y}^2\frac{\partial\widetilde{X}^1}{\partial u} = iu - (-iu) = 2iu \\ [\widetilde{X},\widetilde{Y}]^2 &= \widetilde{X}^1\frac{\partial\widetilde{Y}^2}{\partial z} - \widetilde{Y}^1\frac{\partial\widetilde{X}^2}{\partial z} + \widetilde{X}^2\frac{\partial\widetilde{Y}^2}{\partial u} - \widetilde{Y}^2\frac{\partial\widetilde{X}^2}{\partial u} = u - u = 0.\end{aligned}$$

So $[\widetilde{X},\widetilde{Y}](z,u) = (2iu, 0)$. Back in the coordinates (x,y,θ) this is

$$[X,Y](x,y,\theta) = (-2\sin\theta, 2\cos\theta, 0).$$

(c) Suppose the initial θ is 0, then $[X,Y] = (0,2,0)$. Let us call

$$\phi_Y^{-t}\circ\phi_X^{-t}\circ\phi_Y^t\circ\phi_X^t(x)$$

one parking movement by time t (see the image drawn in the lecture). By exercise 2c: We we get back to x with an error $t^2[X,Y] = (0, 2t^2, 0)$. So we moved in y-direction by $2t^2$. We interpret that in the following way. To park sideways when having space t in the horizontal direction we can move sideways by t^2. For small t, t^2 is very small, so we need to do the parking movement several times if the space t is small.