Exercise Sheet 2

To be handed in until October 4

1. On curvature and torsion of curves in $\mathbb{R}^3$

(a) Compute the scalar curvature k and torsion l at $t = 0$ for the curve

$$t \mapsto (t, at^2, bt^3) \quad a, b \in \mathbb{R}.$$

(b) Show that if a curve γ in $\mathbb{R}^3$ has identically vanishing scalar torsion then γ lies in a plane.

(c) Suppose that a curve γ in $\mathbb{R}^3$ has constant scalar curvature and torsion. Show that γ must be a helix.

Solution:

(a) The curve γ is not parametrized by arc length, so let's recall and derive general formulas for the curvature and torsion for curves in $\mathbb{R}^3$.

Claim. (i) (lecture) For a regular curve γ in $\mathbb{R}^n$ its curvature vector is

$$\kappa = \frac{1}{|\gamma_t|^2}\left(\gamma_{tt} - \left\langle \gamma_{tt}, \frac{\gamma_t}{|\gamma_t|}\right\rangle \frac{\gamma_t}{|\gamma_t|}\right),$$

(ii) In dimension $n = 3$ the scalar curvature k of a regular curve γ is

$$k = \frac{|\gamma_t \times \gamma_{tt}|}{|\gamma_t|^3}$$

and for an ordinary curve (i.e. $k(t) \neq 0$ for all t) its torsion is

$$l = \frac{\langle \gamma_t \times \gamma_{tt}, \gamma_{ttt}\rangle}{|\gamma_t \times \gamma_{tt}|^2}.$$

Proof. (i) The proof was given in the lecture.

(ii) The curvature vector in dimension 3 can be written as

$$\kappa = \frac{|\gamma_t|^2\gamma_{tt} - \langle \gamma_{tt}, \gamma_t\rangle \gamma_t}{|\gamma_t|^4} = \frac{(\gamma_t \times \gamma_{tt}) \times \gamma_t}{|\gamma_t|^4},$$

where we used in the last equality the vector product identity

$$(u \times v) \times w = \langle u, w\rangle v - \langle v, w\rangle u.$$

As $\gamma_t \times \gamma_{tt}$ is orthogonal to γ_t we get

$$k = |\kappa| = \frac{|(\gamma_t \times \gamma_{tt}) \times \gamma_t|}{|\gamma_t|^4} = \frac{|\gamma_t \times \gamma_{tt}||\gamma_t|}{|\gamma_t|^4} = \frac{|\gamma_t \times \gamma_{tt}|}{|\gamma_t|^3}$$

The formula for the normailized vector is

$$N = \frac{\kappa}{k} = \frac{(\gamma_t \times \gamma_{tt}) \times \gamma_t}{|\gamma_t \times \gamma_{tt}||\gamma_t|}.$$

The binormal $B = \tau \times N$ is orthogonal to τ and to N. Looking at the formulas. B must be parallel to $\gamma_t \times \gamma_{tt}$. After normalizing we get

$$B = \frac{\gamma_t \times \gamma_{tt}}{|\gamma_t \times \gamma_{tt}|}.$$

The torsion l is given by

$$\begin{aligned}
l &= \left\langle \frac{dN}{ds}, B \right\rangle = -\left\langle \frac{dB}{ds}, N \right\rangle = -\frac{1}{|\gamma_t|} \left\langle \frac{dB}{dt}, N \right\rangle \\
&= -\frac{1}{|\gamma_t|} \left\langle \frac{(\gamma_{tt} \times \gamma_{tt} + \gamma_t \times \gamma_{ttt})|\gamma_t \times \gamma_{tt}| - \gamma_t \times \gamma_{tt} \frac{d}{dt}(|\gamma_t \times \gamma_{tt}|)}{|\gamma_t \times \gamma_{tt}|^2}, N \right\rangle \\
&= -\frac{1}{|\gamma_t|} \left\langle \frac{\gamma_t \times \gamma_{ttt}}{|\gamma_t \times \gamma_{tt}|}, N \right\rangle \\
&= -\frac{1}{|\gamma_t|} \left\langle \frac{\gamma_t \times \gamma_{ttt}}{|\gamma_t \times \gamma_{tt}|}, \frac{(\gamma_t \times \gamma_{tt}) \times \gamma_t}{|\gamma_t \times \gamma_{tt}||\gamma_t|} \right\rangle \\
&= -\frac{\langle \gamma_t \times \gamma_{ttt}, (\gamma_t \times \gamma_{tt}) \times \gamma_t \rangle}{|\gamma_t|^2 |\gamma_{tt} \times \gamma_t|^2} \\
&= -\frac{\left\langle \gamma_t \times \gamma_{ttt}, |\gamma_t|^2 \gamma_{tt} - \langle \gamma_t, \gamma_{tt} \rangle \gamma_t \right\rangle}{|\gamma_t \times \gamma_{tt}|^2} \\
&= -\frac{\langle \gamma_t \times \gamma_{ttt}, \gamma_{tt} \rangle}{|\gamma_t \times \gamma_{tt}|^2} \\
&= \frac{\langle \gamma_t \times \gamma_{tt}, \gamma_{ttt} \rangle}{|\gamma_t \times \gamma_{tt}|^2},
\end{aligned}$$

where in the last step we used

$$\langle u \times v, w \rangle = -\langle u \times w, v \rangle.$$

So l is measuring the component of the third derivative of γ in direction B.

□

In the example stated in the exercise we have

$$\gamma_t = (1, 2at, 3bt^2) \qquad \gamma_{tt} = (0, 2a, 6bt) \qquad \gamma_{ttt} = (0, 0, 6b).$$

Hence at $t = 0$ we get

$$k = \frac{|\gamma_t \times \gamma_{tt}|}{|\gamma_t|^3} = 2a$$

and

$$l = \frac{\langle \gamma_t \times \gamma_{tt}, \gamma_{ttt} \rangle}{|\gamma_{tt} \times \gamma_t|^2} = \frac{3b}{a}.$$

(b) Recall from the lecture that the vectors $\tau(s), N(s), B(s)$ form an orthonormal basis (ONB) of $\mathbb{R}^3$ for each $s \in [0, L]$. Moreover, they change according to the following rules along the curve

$$\frac{d}{ds}\begin{pmatrix} \tau \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & l \\ 0 & -l & 0 \end{pmatrix}\begin{pmatrix} \tau \\ N \\ B \end{pmatrix}$$

If $l = 0$ then $\frac{dB}{ds} = 0$. So $B(s) = B_0$ is a constant unit vector $B_0 \in \mathbb{R}^3$ for all s. As $N(s)$ and $\tau(s)$ are orthogonal to $B(s) = B_0$ for all s, the vector $\tau(s)$ (also N but not important here) lies in the plane orthogonal to B_0. But as γ is just τ integrated once, also γ must stay in the plane orthogonal to B_0.

(c) We show that the helix has constant curvature k and constant torsion l and then use uniqueness from exercise 2.

Any helix in z-direction in $\mathbb{R}^3$ is given by $\gamma : [c, d] \to \mathbb{R}^3$ with

$$\gamma(t) = (R\cos(t), R\sin(t), mt)$$

for some c, d, m all in $\mathbb{R}$ and $R > 0$. The derivatives are:

$$\begin{aligned} \gamma_t &= (-R\sin(t), R\cos(t), m) \\ \gamma_{tt} &= (-R\cos(t), -R\sin(t), 0) \\ \gamma_{ttt} &= (R\sin(t), -R\cos(t), 0) \end{aligned}$$

Using the formulas derived in part (a) we get

$$k = \frac{R}{R^2 + m^2} \qquad l = \frac{m}{R^2 + m^2}.$$

Given l, k we can determine R and m:

$$k^2 + l^2 = \frac{1}{R^2 + m^2}$$

That means $R = \frac{k}{k^2+l^2}$ and $m = \frac{l}{k^2+l^2}$. Therefore, for each pair of functions (k, l), there is a helix with curvature k and torsion l. Moreover, there is no other curve up to rigid motions of space by exercise 2 with the same (k, l).

2. Curvature and torsion determine a curve in $\mathbb{R}^3$ up to rigid motion

Prove that any given smooth functions $k(s)$, $l(s)$, with $k(s) > 0$ determine a curve in $\mathbb{R}^3$ with curvature $k(s)$ and torsion $l(s)$ (where s is the arclength) that is unique up to rigid motion of space (i.e. a composition of rotations and translations).

Hint: Theorem. (*Existence and uniqueness for ODEs*)
Let $U \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set and let $f : U \to \mathbb{R}^n$ be continuous. Moreover, suppose f is *locally Lipschitz in the second coordinate* i.e. for all $(t_0, y_0) \in U$ there is an open neighbourhood $W \subset U$ of (t_0, y_0) and $M > 0$ such that $|f(t, y_2) - f(t, y_2)| \leq M|y_2 - y_1|$ for all $(t, y_1), (t, y_2) \in W$.

For any $(t_0, y_0) \in U$ consider the ODE system

$$(*) = \begin{cases} \dot{y}(t) = f(t, y(t)) \\ y(t_0) = x_0. \end{cases}$$

Then

i (Existence) There exists a small open interval I containing t_0 and a continuously differentiable function $y : I \to \mathbb{R}^n$ that solves $(*)$.

ii (Uniqueness) Suppose that there are two solutions $y, \tilde{y}$ of $(*)$ defined on intervals I and $\tilde{I}$ respectively. Then $y, \tilde{y}$ agree on the intersection $I \cap \tilde{I}$.

Solution:

Existence: Let us first construct a canonical curve γ in $\mathbb{R}^3$ which is parametrized by arclength with curvature k and torsion l for any given functions k and l with $k(s) > 0$ and defined for $s \in I$, where I is an interval. Denote by (e_1, e_2, e_3) the standard basis of $\mathbb{R}^3$ and let this be the initial condition for (τ, N, B) for the ODE

$$\frac{d}{ds} \begin{pmatrix} \tau \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & l \\ 0 & -l & 0 \end{pmatrix} \begin{pmatrix} \tau \\ N \\ B \end{pmatrix}.$$

By the theorem given in the hint, there is a unique solution of an orthonormal frame (τ, N, B) defined on the interval $s \in I$.

(Technical remark: We can find that solutions exist for any $s \in I$ a priori only defined on a small interval containing s but which agree on the intersection with an interval for another $s' \in I$. So actually a solution exists defined on all of I.)

In particular, given τ we can recover a curve γ by integration and fixing the start value $\gamma(0) = 0$ (assuming $0 \in I$) which has by construction curvature k and torsion l.

Uniqueness: Suppose $\tilde{\gamma} : I \to \mathbb{R}^3$ is another curve that is parametrized by arclength, has curvature k and torsion l. Without loss of generality $0 \in I$. We want to prove that we can send $\tilde{\gamma}$ onto γ by a rigid motion of space in $\mathbb{R}^3$.

There is a unique rigid motion of space A (a rotation composed with a translation) that sends $\tilde{\gamma}(0)$ to $\gamma(0)$, $\tilde{\tau}(0)$ to $\tau(0) = e_1$ and $\tilde{N}(0)$ to $N(0) = e_2$

(and automatically also $\tilde{B}(0)$ to $B(0)$). We will show now that A sends the entire curve $\tilde{\gamma}$ to γ. This finishes the proof of uniqueness.

By definition of A, the curve $A\tilde{\gamma}$ and our canonical curve γ have the same initial conditions for (τ, N, B). To prove that $A\tilde{\gamma} = \gamma$ it is enough to show that $A\tilde{\gamma}$ satisfies the same ODE as γ and then use the uniqueness part of the ODE theorem that is given in the hint.

Informally, we already used in the lecture that the curvature and the torsion of a translated and rotated curve stays the same. Let us also give a more formal argument: As the orthonormal frame for $\tilde{\gamma}$ solves the ODE with given (k, l) we just need to show that the orthonormal frame of $A\tilde{\gamma}$ also solves the ODE and then use uniqueness. Suppose X is the matrix $\tilde{X} = (\tilde{\tau} \mid \tilde{N} \mid \tilde{B})$ and suppose $(\tilde{\tau}, \tilde{N}, \tilde{B})$ solves the ODE, i.e. denoting

$$M(s) = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & l(s) \\ 0 & -l(s) & 0 \end{pmatrix}$$

we assume $\frac{d}{ds}\tilde{X} = \tilde{X}M^T$. But then also $A\tilde{X} = (A\tilde{\tau} \mid A\tilde{N} \mid A\tilde{B})$ satisfies $\frac{d}{ds}A\tilde{X} = A\frac{d}{ds}\tilde{X} = A\tilde{X}M^T$. So $(A\tilde{\tau}, A\tilde{N}, A\tilde{B})$ also solves the ODE. This finishes the proof that $A\tilde{\gamma} = \gamma$.

For the following problems, use the definitions:

$$\begin{aligned} k_1, k_2 &: \text{principal curvatures} \\ H = k_1 + k_2 &: \text{mean curvature} \\ K = k_1 k_2 &: \text{Gauss curvature} \end{aligned}$$

3. Curvatures of some standard surfaces

Compute the curvatures k_1, k_2, H and K for

(a) a sphere of radius R,

(b) a cylinder of radius R.

Solution:

(a) Let M be the sphere in $\mathbb{R}^3$ of radius R centered at the origin. Let us choose $N(p) = \frac{-p}{R}$ as the normal vector of a point p on M. The tangent space T_pM are the vectors lying in the plane orthogonal to $N(p)$. Recall that for $v \in T_pM \setminus \{0\}$ we defined

$$Q_p(v) =< \kappa_\gamma(0), N >,$$

where $\kappa_\gamma(t)$ is the curvature vector of a curve $\gamma : (-\varepsilon, \varepsilon) \to M$ with $\gamma(0) = p$ and $\gamma_t(0) = v$. In the lecture, we have seen that the definition is well-defined, i.e. does not depend on the choice of γ.

To get all directions $v \in T_pM$ with $|v| = 1$ we can choose as the curves the great circles. These are all circles of radius R which have constant curvature $\frac{1}{R}$ and curvature vector pointing everywhere in the direction of the origin, so pointing to the same direction as the normal vector field N. Indeed, we have already seen the computation of a circle in the plane $\mathbb{R}^2$ and computing the unit tangent vectors and curvature vectors commutes with rotation and translation of the curve.

So we proved that $Q_p(v) = \frac{1}{R}$ for all unit tangent vectors $v \in T_pM$ and hence

$$k_1 = \min_{v \in T_pM, |v|=1} = \frac{1}{R}, \qquad k_2 = \max_{v \in T_pM, |v|=1} = \frac{1}{R}.$$

(b) Let M be the cylinder in $\mathbb{R}^3$ of radius R, that is

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid y^2 + z^2 = R^2\}.$$

Because of obvious rotational and translational symmetry, it is enough to compute the curvature for only one point, let's say for $p = (0, 0, R)$. The normal vector is given by $p = (0, 0, -1)$. The tangent plane T_pM at p is given $z = 0$, so all unit tangent vectors at p are of the form $v_\theta = (\cos\theta, \sin\theta, 0)$ for some $\theta \in [0, 2\pi)$. To compute $Q(v_\theta)$ let us look at the curve given by intersecting M with the plane orthogonal to $v_{\theta+\frac{\pi}{2}}$. For $\theta = 0$ we get two straight lines but only one passing through p. Straight lines have curvature 0, so $Q(v_0) = 0$. For the other angles $\theta \in (0, 2\pi)$ we get an ellipse centered at the origin. One axis is of length $a = R$ the other axis of length $b = \frac{R}{\sin\theta}$. So we need to compute the curvature of an ellipse γ of the form $t \mapsto (a\cos(t), b\sin(t))$ at $t = 0$. We have

$$\gamma_t(t) = (-a\sin(t), b\cos(t)), \gamma_{tt}(t) = (-a\cos(t), -b\sin(t))$$

So using the formula for the curvature of a curve that is not parametrized by arclength we get

$$\begin{aligned} k &= \frac{|\gamma_t \times \gamma_{tt}|}{|\gamma_t|^3} \\ &= \frac{|(-a\sin(t), b\cos(t), 0) \times (-a\cos(t), -b\sin(t), 0)|}{|(-a\sin(t), b\cos(t), 0)|^3} \\ &= \frac{|(0, 0, ab)|}{(a^2\sin^2(t) + b^2\cos^2(t))^{3/2}} \\ &= \frac{ab}{(a^2\sin^2(t) + b^2\cos^2(t))^{3/2}}. \end{aligned}$$

For $t = 0$ this is $\kappa(0) = \frac{a}{b^2}$. In our case $a = R$ and $b = \frac{R}{\sin\theta}$ and as the direction of the curvature vector of the ellipse is parallel to the normal N we get

$$Q(v_\theta) = \kappa(0) = \frac{\sin^2\theta}{R}$$

for $\theta \in [0, 2\pi)$. So the principal curvatures are $k_1 = 0$ in direction $(\pm 1, 0, 0)$ and $k_2 = \frac{1}{R}$ in direction $(0, \pm 1, 0)$.

4. Curvatures of surfaces of revolution

A surface of revolution in $\mathbb{R}^3$ is defined by

$$M = M_f := \{(x, y, z) \in I \times \mathbb{R}^2 \,|\, f(x) = \sqrt{y^2 + z^2}\} \subset \mathbb{R}^3,$$

where $f : I \to \mathbb{R}$ is a smooth positive function, I an interval. The curve γ given by $y = f(x)$ in the plane $\mathbb{R}^2$ is called the *generator* of M. Find k_1, k_2, H and K for M.

Hint: You can use without proof (but think about it) that the principal directions of a surface of revolution are in the direction tangent to γ and normal to γ. Useful notation: $r = \sqrt{y^2 + z^2}$ and $re^{i\theta} = y + iz = (y, z)$.

Solution:

Using the rotational symmetry of M it is enough to compute the curvature at a point $p = (x, 0, f(x))$. Denote e_x the tangent vector to the surface in x direction and e_θ the tangent vector to the surface in the direction of the rotation, i.e.

$$e_x(p) = \frac{(1, 0, f_x(x))}{\sqrt{1 + f_x(x)^2}}, \qquad e_\theta(p) = (0, 1, 0).$$

The tangent plane T_pM is spanned by e_x and e_θ. Let us choose the outward-pointing normal vector to the surface at p given by

$$N(p) = \frac{(-f_x(x), 0, 1)}{\sqrt{1 + f_x(x)^2}}.$$

Denote by k_x the curvature of the surface of M in direction e_x. This is just the curvature of the graph of the function f as it parametrizes a curve with direction e_x at p. We have seen the formula of the curvature of a graph of a function $f : I \to \mathbb{R}$ in the lecture. So we get

$$k_x(p) = \frac{f_{xx}(x)}{(1 + f_x^2(x))^{3/2}}.$$

Denote k_θ the curvature in the direction of the rotation. As the surface of revolution intersected with a plane with constant x-coordinate is just a circle γ of radius $f(x)$ we get a scalar curvature $\frac{1}{f(x)}$. However, the normal to M is not parallel to the curvature vector of this circle. Actually,

$$k_\theta(p) = \left\langle \frac{(0,0,-1)}{f(x)}, \frac{(-f_x(x),0,1)}{\sqrt{1+f_x(x)^2}} \right\rangle = \frac{-1}{f(x)\sqrt{1+f_x(x)^2}}.$$

In the lecture, we proved that k_θ, k_x are actually the principal curvatures of M and e_θ, e_x are the principal directions using that reflecting M across the xz-plane in $\mathbb{R}^3$ sends M to M.

Therefore, the curvatures are

$$H = k_\theta + k_x = \frac{1}{\sqrt{1+f_x^2}} \left(\frac{f_{xx}}{1+f_x^2} - \frac{1}{f} \right), \qquad K = k_\theta k_x = -\frac{f_{xx}}{f(1+f_x^2)^2}.$$