Exercise Sheet 3

To be handed in until October 11

1. The Catenoid

Compute k_1, k_2, H and K for the catenoid.

Solution:

The catenoid is the surface of revolution for the generating curve $y = f(x)$, where $f(x) = \cosh x$. So we can apply the formulas derived in exercise 4 from sheet 2. The derivatives of f are:

$$f_x(x) = \sinh x, \qquad f_{xx} = \cosh x$$

For $p = (x, y, z)$ we get in the curve direction e_x curvature

$$k_x(p) = \frac{f_{xx}(x)}{(1 + f_x^2(x))^{3/2}} = \frac{\cosh x}{(1 + \sinh^2 x)^{3/2}} = \frac{1}{\cosh^2 x}$$

since $\cosh^2 x - \sinh^2 x = 1$.

In the rotation direction e_θ we get curvature

$$k_\theta(p) = \frac{-1}{f(x)\sqrt{1 + f_x(x)^2}} = \frac{-1}{\cosh(x)\sqrt{1 + \sinh^2}} = -\frac{1}{\cosh^2 x}.$$

In the lecture, we have seen that for surfaces of revolution, e_x and e_θ are the principal directions. Hence k_x and k_θ are the principal curvatures.

Therefore, as

$$H = k_x + k_\theta = 0,$$

the catenoid is a minimal surface. Its Gauss curvature is

$$K(p) = \frac{-1}{\cosh^4 x}.$$

2. The Helicoid

Compute k_1, k_2, H and K for the helicoid.

Solution:

The helicoid is the image of $\Psi : \mathbb{R}^2 \to \mathbb{R}^3$ given by the parametrization

$$(r, t) \mapsto (r \cos t, r \sin t, mt).$$

Claim. Let us derive the formula for the 2nd fundamental form for a regular parametrization Ψ of a surface M. Regular here means that the vectors $\Psi_r(r,t), \Psi_t(r,t)$ are linearly independent for all r,t, so in particular span the tangent space T_pM at $p = \Psi(r,t)$. We claim that with respect to this basis the 2nd fundamental form is given by the following symmetric matrix

$$\begin{pmatrix} \langle \Psi_{rr}, N\rangle & \langle \Psi_{rt}, N\rangle \\ \langle \Psi_{tr}, N\rangle & \langle \Psi_{tt}, N\rangle \end{pmatrix}.$$

Proof. The normal vector at $p \in M$ is

$$N = \frac{\Psi_r \times \Psi_t}{|\Psi_r \times \Psi_t|}.$$

The 2nd fundamental form is a bilinear form $A : T_pM \times T_pM \to \mathbb{R}$. So it is enough to know it for a basis of T_pM. Recall from the lecture that for $X, Y \in T_pM$ we have $A(X,Y) = -\langle D_X N, Y\rangle$. Let us apply this formula to the basis Ψ_r, Ψ_t.

$$D_{\Psi_r}N = (N \circ \Psi)_r = \frac{(\Psi_{rr} \times \Psi_t + \Psi_r \times \Psi_{tr})}{|\Psi_r \times \Psi_t|} - \frac{(\Psi_r \times \Psi_t)\frac{d}{dr}|\Psi_r \times \Psi_t|}{|\Psi_r \times \Psi_t|^2}$$

Analogeously we get a formula for $D_{\Psi_t}N$. Note that the $\Psi_r \times \Psi_t$ is orthogonal to Ψ_r and to Ψ_t so the last term will cancel, once we take the scalar product with Ψ_r or Ψ_t. Indeed,

$$\begin{aligned} A(\Psi_r, \Psi_r) = -\langle D_{\Psi_r}N, \Psi_r\rangle &= -\left\langle \frac{\Psi_{rr} \times \Psi_t + \Psi_r \times \Psi_{tr}}{|\Psi_r \times \Psi_t|}, \Psi_r \right\rangle \\ &= -\frac{\langle \Psi_{rr} \times \Psi_t, \Psi_r\rangle}{|\Psi_r \times \Psi_t|} \\ &= -\frac{\det(\Psi_r|\Psi_{rr}|\Psi_t)}{|\Psi_r \times \Psi_t|} \\ &= \frac{\det(\Psi_{rr}|\Psi_r|\Psi_t)}{|\Psi_r \times \Psi_t|} \\ &= \frac{\langle \Psi_{rr}, \Psi_r \times \Psi_t\rangle}{|\Psi_r \times \Psi_t|} \\ &= \langle \Psi_{rr}, N\rangle \end{aligned}$$

$$\begin{aligned}
A(\Psi_t, \Psi_t) = -\langle D_{\Psi_t} N, \Psi_t \rangle &= -\left\langle \frac{\Psi_{rt} \times \Psi_t + \Psi_r \times \Psi_{tt}}{|\Psi_r \times \Psi_t|}, \Psi_t \right\rangle \\
&= -\frac{\langle \Psi_r \times \Psi_{tt}, \Psi_t \rangle}{|\Psi_r \times \Psi_t|} \\
&= -\frac{\det(\Psi_t | \Psi_r | \Psi_{tt})}{|\Psi_r \times \Psi_t|} \\
&= \frac{\det(\Psi_{tt} | \Psi_r | \Psi_t)}{|\Psi_r \times \Psi_t|} \\
&= \frac{\det(\Psi_{tt} | \Psi_r | \Psi_t)}{|\Psi_r \times \Psi_t|} \\
&= \langle \Psi_{tt}, N \rangle
\end{aligned}$$

$$\begin{aligned}
A(\Psi_r, \Psi_t) = -\langle D_{\Psi_r} N, \Psi_t \rangle &= \left\langle \frac{\Psi_{rr} \times \Psi_t + \Psi_r \times \Psi_{tr}}{|\Psi_r \times \Psi_t|}, \Psi_t \right\rangle \\
&= -\frac{\langle \Psi_r \times \Psi_{tr}, \Psi_t \rangle}{|\Psi_r \times \Psi_t|} \\
&= -\frac{\det(\Psi_t | \Psi_r | \Psi_{tr})}{|\Psi_r \times \Psi_t|} \\
&= \frac{\det(\Psi_{tr} | \Psi_r | \Psi_t)}{|\Psi_r \times \Psi_t|} \\
&= \langle \Psi_{tr}, N \rangle
\end{aligned}$$

□

The helicoid is the image of $\Psi : \mathbb{R}^2 \to \mathbb{R}^3$ given by the parametrization

$$(r, t) \mapsto (r \cos t, r \sin t, mt).$$

The tangent plane at a given point $p = \Psi(r, t) \in \mathbb{R}^3$ is spanned by $\Psi_r(r, t)$ and $\Psi_t(r, t)$, which are

$$\Psi_r(r, t) = (\cos t, \sin t, 0) \qquad \Psi_t(r, t) = (-r \sin t, r \cos t, m).$$

So the normal vector to the surface is

$$N = \frac{\Psi_r \times \Psi_t}{|\Psi_r \times \Psi_t|} = \frac{(m \sin(t), -m \cos(t), r)}{\sqrt{r^2 + m^2}}.$$

To compute the 2nd fundamental form, we compute the second derivatives:

$$\begin{aligned}
\Psi_{rr}(r, t) &= (0, 0, 0) \\
\Psi_{rt}(r, t) &= (-\sin t, \cos t, 0) \\
\Psi_{tt}(r, t) &= (-r \cos t, -r \sin t, 0)
\end{aligned}$$

Hence in the basis $\Psi_r(r,t), \Psi_t(r,t)$ of T_pM the 2nd fundamental form is

$$\begin{pmatrix} \langle \Psi_{rr}, N\rangle & \langle \Psi_{rt}, N\rangle \\ \langle \Psi_{tr}, N\rangle & \langle \Psi_{tt}, N\rangle \end{pmatrix} = \begin{pmatrix} 0 & \frac{-m}{\sqrt{r^2+m^2}} \\ \frac{-m}{\sqrt{r^2+m^2}} & 0 \end{pmatrix}.$$

We can not yet read the principal curvatures from this matrix, only in case the matrix is with respect to an orthonormal basis. However, in our situation, Ψ_r and Ψ_t are already orthogonal, and Ψ_r is normalized. With respect to the ONB $e_1 = \Psi_r$ and $e_2 = \frac{\Psi_t}{|\Psi_t|}$, the 2nd fundamental form is represented by matrix

$$\begin{pmatrix} 0 & \frac{-m}{r^2+m^2} \\ \frac{-m}{r^2+m^2} & 0 \end{pmatrix}.$$

as both entries get scaled by $\frac{1}{|\Psi_t|} = \frac{1}{\sqrt{r^2+m^2}}$.

Note that the entries on the diagonal are equal to the curvature of the helix that we computed in sheet 2 exercise 1c. The curvatures are

$$\begin{aligned} H &= \operatorname{tr}(A) = 0, \\ K &= \det A = -\frac{m^2}{(r^2+m^2)^2}, \\ -k_1 = k_2 &= \frac{m}{r^2+m^2}. \end{aligned}$$

3. Compact surfaces have positive K

Let M be a compact surface in $\mathbb{R}^3$. Prove that there is a point p in M such that $K(p) > 0$.

Solution:

Let $p \in M$ be the point in M with maximal Euclidean norm (exists as M is compact). Then M is contained in a ball B of radius $R = |p|$ centered at the origin. Let S be the sphere of radius R centered at the origin (the boundary of the ball). Note that also by definition of R that $p \in S$. Moreover, the tangent space T_pM at p is the same as T_pS. Indeed, as p was the point in M with maximal norm, in no direction in M the norm can increase. In other words, as p is the point of maximal norm, it is in particular a critical point of the norm function $|\cdot| : M \to [0, \infty)$.

Moreover, as M is contained in the ball B and $p \in S$ is on the boundary, the curvature of curves in M going through p must at least curve as much as curves in the sphere S at p to stay inside B. As the the normal of S and the normal of M at p agree we have $k_1(p), k_2(p) \geq \frac{1}{R}$ (or $k_1, k_2 \leq \frac{-1}{R}$ for outward pointing normal). In any case,

$$K(p) = k_1(p)k_2(p) \geq \frac{1}{R^2}.$$

4. Vanishing 2nd fundamental form implies planar surface

Suppose M is a connected surface in $\mathbb{R}^3$ with 2nd fundamental form A vanishing everywhere. Show that M is contained in a plane.

Solution:

Let N be the normal to the surface M. We want to show that N is constant. Equivalently, as M is connected, it is enough to show that the directional derivative $D_X N(p) = 0$ for each point $p \in M$ and direction $X \in T_pM$. Note that $D_X N(p) \in T_pM$ is a vector. To show that is zero, we test it against every vector $Y \in T_pM$. But using the formula from the lecture

$$\langle D_X N(p), Y \rangle = A(X, Y) = 0,$$

we get the claim as $A = 0$ by assumption. This shows that the normal $N(p)$ is the same vector for all p, and hence M lies in a plane orthogonal to it.