Exercise Sheet 4

To be handed in until October 18

1. Helicoid and catenoid are locally isometric

(a) Find a local isometry

$$\varphi : \text{helicoid} \to \text{catenoid}.$$

(b) Verify that φ preserves K. What does it do to the principle curvatures and principle directions?

(c*) Show that there is a continuous family of minimal surfaces deforming the helicoid into the catenoid.

Solution:

(a) Let $a \in \mathbb{R}$. Define the following two parametrizations $\Psi^{C,a}, \Psi^{H,a} : U \to \mathbb{R}^3$ where

$$\Psi^{C,a}(u,v) = (a\cosh u\cos v, a\cosh u\sin v, au)$$
$$\Psi^{H,a}(u,v) = (a\sinh u\cos v, a\sinh u\sin v, av)$$

for $U = \{(u,v) \in \mathbb{R}^2 \mid -\pi < y < \pi\}$. Note that Ψ^C_a is a parametrization of a catenoid as the surface of revolution of the curve $x \mapsto a\cosh\frac{x}{a}$ (for $a \neq 0$) around the z-axis. On the other hand, Ψ^C_a is a parametrization of a helicoid.

To compare the two parametrizations, let's compute how they depend on the parameters u, v:

$$\Psi^{C,a}_u = (a\sinh u\cos v, a\sinh u\sin v, a)$$
$$\Psi^{C,a}_v = (-a\cosh u\sin v, a\cosh u\cos v, 0)$$
$$\Psi^{H,a}_u = (a\cosh u\cos v, a\cosh u\sin v, 0)$$
$$\Psi^{H,a}_v = (-a\sinh u\sin v, a\sinh u\cos v, a)$$

The first fundamental form of the catenoid in the basis $\Psi^{C,a}_u$ and $\Psi^{C,a}_v$ is given by

$$\begin{pmatrix} |\Psi^{C,a}_u|^2 & \langle\Psi^{C,a}_u, \Psi^{C,a}_v\rangle \\ \langle\Psi^{C,a}_u, \Psi^{C,a}_v\rangle & |\Psi^{C,a}_v|^2 \end{pmatrix} = \begin{pmatrix} a^2(\sinh^2 u + 1) & 0 \\ 0 & a^2\cosh^2 u \end{pmatrix}.$$

The first fundamental form of the helicoid in the basis $\Psi_u^{H,a}$ and $\Psi_v^{H,a}$ is given by

$$\begin{pmatrix} |\Psi_u^{H,a}|^2 & \langle \Psi_u^{H,a}, \Psi_v^{H,a}\rangle \\ \langle \Psi_u^{H,a}, \Psi_v^{H,a}\rangle & |\Psi_v^{H,a}|^2 \end{pmatrix} = \begin{pmatrix} a^2\cosh^2 u & 0 \\ 0 & a^2(\sinh^2 u + 1) \end{pmatrix}.$$

Since $\sinh^2 u + 1 = \cosh^2 u$ the two parametrizations have the same first fundamental form, hence

$$\phi := \Psi^{C,a} \circ (\Psi^{H,a})^{-1} \, : \, \text{helicoid} \to \text{catenoid}$$

defines a local isometry for each a.

(b) To get the curvatures we need to compute the 2nd fundamental forms and the normals. Let us compute the second derivatives.

$$\begin{aligned}
\Psi_{uu}^{C,a} &= (a\cosh u \cos v, a\cosh u \sin v, 0) \\
\Psi_{uv}^{C,a} &= (-a\sinh u \sin v, a\sinh u \cos v, 0) \\
\Psi_{vv}^{C,a} &= (-a\cosh u \cos v, -a\cosh u \sin v, 0) \\
\Psi_{uu}^{H,a} &= (a\sinh u \cos v, a\sinh u \sin v, 0) \\
\Psi_{uv}^{H,a} &= (-a\cosh u \sin v, a\cosh u \cos v, 0) \\
\Psi_{vv}^{H,a} &= (-a\sinh u \cos v, -a\sinh u \sin v, a).
\end{aligned}$$

The normal is

$$\begin{aligned}
N^{C,a} = \frac{\Psi_u^{C,a} \times \Psi_v^{C,a}}{|\Psi_u^{C,a}||\Psi_v^{C,a}|} &= \frac{(-a^2\cosh u \cos v, -a^2\cosh u \sin v, a^2\sinh u \cosh u)}{a^2\cosh^2 u} \\
&= \frac{(-\cos v, -\sin v, \sinh u)}{\cosh u}
\end{aligned}$$

for the catenoid and

$$\begin{aligned}
N^{H,a} = \frac{\Psi_u^{H,a} \times \Psi_v^{H,a}}{|\Psi_u^{H,a}||\Psi_v^{H,a}|} &= \frac{(a^2\cosh u \sin v, -a^2\cosh p[u \cos v, a^2\cosh u \sinh u)}{a^2\cosh^2 u} \\
&= \frac{(\sin v, -\cos v, \sinh u)}{\cosh u}
\end{aligned}$$

for the helicoid.

As $|\Psi_u^{C,a}| = |\Psi_v^{C,a}| = a\cosh u$ and using the argument from sheet 3 exercise 2, the formula for second fundamental form of the catenoid with respect to the orthonormal basis $\frac{\Psi_u^{C,a}}{|\Psi_u^{C,a}|}, \frac{\Psi_v^{C,a}}{|\Psi_v^{C,a}|}$ is

$$\begin{aligned}
&\frac{1}{a^2\cosh^2 u}\begin{pmatrix} \langle \Psi_{uu}^{C,a}, N\rangle & \langle \Psi_{uv}^{C,a}, N\rangle \\ \langle \Psi_{vu}^{C,a}, N\rangle & \langle \Psi_{vv}^{C,a}, N\rangle \end{pmatrix} \\
&\qquad = \frac{1}{a^2\cosh^2 u}\begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}.
\end{aligned}$$

So taking the trace we get $H^{C,a} = 0$ and taking the determinant we get

$$K^{C,a} = -\frac{1}{a^2 \cosh^4 u}.$$

Similarly as $|\Psi_u^{H,a}| = |\Psi_v^{H,a}| = a \cosh u$ the second fundamental form of the helicoid with respect to the basis $\frac{\Psi_u^{H,a}}{|\Psi_u^{H,a}|}, \frac{\Psi_v^{H,a}}{|\Psi_v^{H,a}|}$ is

$$\frac{1}{a^2 \cosh^2 u} \begin{pmatrix} 0 & -a \\ -a & 0 \end{pmatrix}.$$

So $H^{H,a} = 0$ and

$$K^{H,a} = -\frac{1}{a^2 \cosh^4 u}.$$

Hence in both cases $k_1 = -\frac{1}{a \cosh^2 u}, k_2 = \frac{1}{a \cosh^2 u}$. The principal directions for the catenoid are

$$e_1^{C,a} = \frac{\Psi_u^{C,a}}{|\Psi_u^{C,a}|}, \qquad e_2^{C,a} = \frac{\Psi_v^{C,a}}{|\Psi_v^{C,a}|}$$

whereas the principal directions for the helicoid are

$$\frac{\Psi_u^{H,a}}{|\Psi_u^{H,a}|}, \qquad \frac{\Psi_v^{H,a}}{|\Psi_v^{H,a}|}$$

rotated by $\pi/2$.

(c) A regular conformal parametrization is a parametrization $\Psi : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ such that the two directional derivatives Ψ_u, Ψ_v satisfy:

$$|\Psi_u| = |\Psi_v| \neq 0 \qquad \text{and} \qquad \Psi_u \perp \Psi_v.$$

Claim. (i) Suppose Ψ is a regular conformal parametrization. Then $\Psi(U)$ is minimal iff $\Psi_{uu} + \Psi_{vv} = 0$.

(ii) Suppose $\Psi^C, \Psi^H : U \to \mathbb{R}^3$ are conformal regular parametrizations and

$$\Psi_u^C \perp \Psi_v^H, \qquad \Psi_u^H \perp \Psi_v^C \qquad \text{and} \qquad \langle \Psi_u^C, \Psi_u^H \rangle = \langle \Psi_v^C, \Psi_v^H \rangle.$$

Then

$$\Psi^t(u,v) = \cos t \, \Psi^C(u,v) + \sin t \, \Psi^H(u,v)$$

is a regular conformal parametrization for every $t \in \mathbb{R}$.

(iii) In the setting of (ii) and if Ψ^C and Ψ^H both define minimal surfaces prove that also Ψ^t parametrizes a minimal surface for every t.

Proof. (i) By assumption $\Psi_u(u,v)$ and $\Psi_u(u,v)$ are an orthogonal basis of T_pM for $p = \Psi(u,v)$. As in exercise 2 sheet 3 we know that the 2nd fundamental form in orthonormal coordinates $e_1 = \frac{\Psi_u}{|\Psi_u|}, e_2 = \frac{\Psi_v}{|\Psi_v|}$ is represented by the symmetric matrix

$$\begin{pmatrix} \frac{\langle \Psi_{uu}, N\rangle}{|\Psi_u|^2} & \frac{\langle \Psi_{uv}, N\rangle}{|\Psi_u||\Psi_v|} \\ \frac{\langle \Psi_{vu}, N\rangle}{|\Psi_u||\Psi_v|} & \frac{\langle \Psi_{vv}, N\rangle}{|\Psi_v|^2} \end{pmatrix}$$

where $N = e_1 \times e_2$. The mean curvature is the trace of this matrix:

$$H = \frac{\langle \Psi_{uu}, N\rangle}{|\Psi_u|^2} + \frac{\langle \Psi_{vv}, N\rangle}{|\Psi_v|^2}.$$

As $|\Psi_u| = |\Psi_v|$ we conclude that $H = 0$ iff $\langle \Psi_{uu} + \Psi_{vv}, N\rangle = 0$. To get the claimed statement, we need to prove that $\Psi_{uu} + \Psi_{vv}$ is parallel to N, i.e. orthogonal to both Ψ_u and Ψ_v. Indeed: Taking the derivatives of

$$\langle \Psi_u, \Psi_u\rangle = \langle \Psi_v, \Psi_v\rangle \qquad \langle \Psi_u, \Psi_v\rangle = 0,$$

we get equations

$$\langle \Psi_{uu}, \Psi_u\rangle = \langle \Psi_{uv}, \Psi_v\rangle = -\langle \Psi_u, \Psi_{vv}\rangle.$$

So $\langle \Psi_{uu} + \Psi_{vv}, \Psi_u\rangle = 0$. Similarly $\langle \Psi_{uu} + \Psi_{vv}, \Psi_v\rangle = 0$.

(ii) The directional derivatives are

$$\begin{aligned} \Psi_u^t &= \cos t\, \Psi_u^C + \sin t\, \Psi_u^H \\ \Psi_v^t &= \cos t\, \Psi_v^C + \sin t\, \Psi_v^H \end{aligned}$$

So we get

$$\begin{aligned} |\Psi_u^t|^2 &= \cos^2 t |\Psi_u^C|^2 + 2\sin t \cos t \langle \Psi_u^C, \Psi_u^H\rangle + \sin^2 t |\Psi_u^H|^2 \\ &= \cos^2 t |\Psi_v^C|^2 + 2\sin t \cos t \langle \Psi_v^C, \Psi_v^H\rangle + \sin^2 t |\Psi_v^H|^2 = |\Psi_v^t|^2 \end{aligned}$$

and

$$\langle \Psi_u^t, \Psi_v^t\rangle = \cos^2 t \langle \Psi_u^C, \Psi_v^C\rangle + \sin t \cos t \left(\langle \Psi_u^C, \Psi_v^H\rangle + \langle \Psi_u^H, \Psi_v^C\rangle\right) + \sin^2 t \langle \Psi_u^H, \Psi_v^H\rangle = 0.$$

(iii) Using the characterization of minimal surfaces from (i) twice we get

$$\begin{aligned} \Psi_{uu}^t + \Psi_{vv}^t &= \cos t\, \Psi_{uu}^C + \sin t\, \Psi_{uu}^H + \cos t\, \Psi_{vv}^C + \sin t\, \Psi_{vv}^H \\ &= \cos t (\Psi_{uu}^C + \Psi_{vv}^C) + \sin t (\Psi_{uu}^H + \Psi_{vv}^H) = 0, \end{aligned}$$

and hence Ψ^t parametrizes a minimal surface.

□

We see that $\Psi^{C,a}$ and $\Psi^{C,a}$ conformal regular parametrizations for each a. Moreover, $\Psi^{C,a}_u \perp \Psi^{H,a}_v$ and $\Psi^{C,a}_v \perp \Psi^{H,a}_u$.

As the catenoid and the helicoid are minimal surfaces we found a 2-parameter family of minimal surface given as the image of $\Psi^{a,t}$.

2. More on isometries

A local isometry between surfaces in $\mathbb{R}^3$ preserves the Gauss curvature K but normally not k_1, k_2, H, or the principal directions of curvature. So the situation in exercise 1 was special in this respect.

(a) Compute K, k_1, k_2, H of a cone.

(b) Show that the cone (minus the vertex) is locally isometric to the plane.

(c) Is there a global isometry between the cone (minus the vertex) and the plane minus a point?

Solution:

(a) Let 2ϕ be the angle of the cone at the vertex. That is the cone is the surface of revolution for the curve $y = f(x) = x\tan\phi$. As $f_x = \tan\phi$ and $f_{xx} = 0$ we get using the formulas from exercise 4 sheet 2 that the principal curvatures are:

$$\begin{aligned} k_x &= \frac{f_{xx}(x)}{(1+f_x^2(x))^{3/2}} = 0 \\ k_\theta &= \frac{-1}{f(x)\sqrt{1+f_x(x)^2}} = \frac{-1}{x\tan\phi\sqrt{1+\tan\phi^2}} \\ &= \frac{-\cos\phi}{x\sin\phi\sqrt{\frac{\cos^2\phi+\sin^2\phi}{\cos^2\phi}}} = \frac{-\cos^2\phi}{x\sin\phi}. \end{aligned}$$

So $K = 0$ and $H = k_\theta$.

(b) The cone with angle 2ϕ at the vertex can be obtained by folding a sector of the disk of angle $2\pi\sin\phi$ along the two straight segments. Denote $U = \{(u,v) \in \mathbb{R}^2 \mid u \in (0,\infty), -\pi\sin\phi < v \leq \pi\sin\phi\}$. The relevant sector of the disk is the image of $\Psi^S : U \to \mathbb{R}^3$ given by $(u,v) \mapsto (u\cos v, u\sin v, 0)$. The first fundamental form in the basis Ψ^S_u and Ψ^S_v is given by

$$\begin{pmatrix} |\Psi^S_u|^2 & \langle \Psi^S_u, \Psi^S_v\rangle \\ \langle \Psi^S_u, \Psi^S_v\rangle & |\Psi^S_v|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u^2 \end{pmatrix}.$$

The cone is the image of the map $\Psi^C : U \to \mathbb{R}^3$ given by

$$(u, v) \mapsto \left(u \cos\phi, u \sin\phi \cos\left(\frac{v}{\sin\phi}\right), u \sin\phi \sin\left(\frac{v}{\sin\phi}\right)\right).$$

The tangent vectors for the cone $\Psi^C(U)$ in the parametrization directions are

$$\Psi_u^C(u, v) = \left(\cos\phi, \sin\phi \cos\left(\frac{v}{\sin\phi}\right), \sin\phi \sin\left(\frac{v}{\sin\phi}\right)\right)$$
$$\Psi_v^C(u, v) = \left(0, -u \sin\left(\frac{v}{\sin\phi}\right), u \cos\left(\frac{v}{\sin\phi}\right)\right)$$

The first fundamental form in the basis Ψ_u and Ψ_v is given by

$$\begin{pmatrix} |\Psi_u|^2 & \langle \Psi_u, \Psi_v \rangle \\ \langle \Psi_u, \Psi_v \rangle & |\Psi_v|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u^2 \end{pmatrix}.$$

As the representations of the first fundamental form agree, the two surfaces are locally isometric by the map

$$\phi := \Psi^S \circ (\Psi^C)^{-1} \,:\, \text{cone} \to \text{subset of the plane}\,.$$

(c) There is no global isometry. Otherwise, there would also be a global isometry from the plane to the cone (we can just add back the deleted point and define the distance to this point as the Euclidean distance in $\mathbb{R}^3$. A global isometry needs to extend. The points of distance 1 from the vertex in the cone and the points of distance 1 from the origin in the plane should get mapped onto each other by a global isometry. Both are a curve (a circle) but they do not have the same length. So no global isometry can exist.

3. The Pseudosphere

The pseudosphere is the surface of revolution for the curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ parametrized by

$$t \mapsto (t - \tanh t, \operatorname{sech} t),$$

where $\tanh t = \frac{\sinh t}{\cosh t}$ and $\operatorname{sech} t = \frac{1}{\cosh t}$. Compute K for the pseudosphere.

Solution:

The curve γ is defined by a parametrization, not explicitly as $y = f(x)$ as in the formulas derived in exercise 4 from sheet 2. We still can compute the necessary derivatives without solving explicitly for the y-coordinate. Let $x(t) = t - \tanh t$ and $y(t) = \operatorname{sech} t$. Note that

$$f_x = y_x = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = y_t t_x.$$

Using $\frac{d}{dt}\tanh t = 1 - \cosh^2 t$ we get

$$x_t = 1 - \frac{1}{\cosh^2 t} = \frac{\cosh^2 t - 1}{\cosh^2 t} = \frac{\sinh^2 t}{\cosh^2 t}, \qquad y_t = -\frac{\sinh t}{\cosh^2 t}.$$

We get the reversed derivatives

$$t_x = \frac{1}{x_t} = \frac{\cosh^2 t}{\sinh^2 t}, \qquad t_y = \frac{1}{y_t} = -\frac{\cosh^2 t}{\sinh t}.$$

This now lets us compute the derivative

$$y_x = y_t t_x = -\frac{\sinh t}{\cosh^2 t}\frac{\cosh^2 t}{\sinh^2 t} = -\frac{1}{\sinh t}.$$

and also the second derivative

$$y_{xx} = (y_x)_t t_x = \frac{\cosh t}{\sinh^2 t}\frac{\cosh^2 t}{\sinh^2 t} = \frac{\cosh^3 t}{\sinh^4 t}.$$

Using

$$(1 + y_x^2)^2 = \left(1 + \frac{1}{\sinh^2 t}\right)^2 = \left(\frac{\sinh^2 t + 1}{\sinh^2 t}\right)^2 = \left(\frac{\cosh^2 t}{\sinh^2 t}\right)^2 = \frac{\cosh^4 t}{\sinh^4 t}$$

in the formula for the Gauss curvature K that we derived in sheet 2 exercise 4 we get

$$K = -\frac{y_{xx}}{y(1 + y_x^2)^2} = -\frac{\cosh^3 t}{\sinh^4 t}\frac{1}{\frac{1}{\cosh t}}\frac{\sinh^4 t}{\cosh^4 t} = -1.$$

So the pseudosphere is of constant negative Gauss curvature -1. That is why it deserves the name pseudosphere in contrast to the sphere which has constant positive curvature 1.

4. Shortest path in $\mathbb{R}^n$

Prove that a straight line in $\mathbb{R}^n$ is the shortest path between two given points.

Solution:

As rotation and translation do not change the distance of two points, it is enough to show the claim for two points that have coordinates $p = (x, 0, \cdots, 0)$

and $\tilde{p} = (\tilde{x}, 0, \cdots, 0)$ in $\mathbb{R}^n$. Also assume $x < \tilde{x}$. The length of a straight line connecting these two points is $\tilde{x} - x$. So we want to show that any curve in $\mathbb{R}^n$ connecting these two points has at least length $\tilde{x} - x$. Let γ be an arbitrary curve with $\gamma(0) = p, \gamma(1) = \tilde{p}$. Write $\gamma(t) = (\gamma^1(t), \ldots, \gamma^n(t))$. Then

$$\text{length}(\gamma) = \int_0^1 |\gamma_t(t)| \, dt \geq \int_0^1 |\gamma_t^1(t)| \, dt \geq \int_0^1 \gamma_t^1(t) \, dt = \tilde{x} - x.$$