Exercise Sheet 5

To be handed in until October 25

1. Two Atlases on the Sphere

(a) Give an atlas of $2n+2$ charts on S^n that are graphs.

(b) Give an atlas of 2 charts on S^n that are given by stereographic projections.

Solution:

Recall $S^n = \{x = (x^0, \dots, x^n) \in \mathbb{R}^{n+1} \mid |x| = 1\}$.

(a) For $j = 0, \dots, n$, denote by

$$U_j^+ = \{x \in S^n \mid x^j > 0\}, \qquad U_j^- = \{x \in S^n \mid x^j < 0\}$$

the hemispheres with respect to the coordinate j. The sets $U_j^\pm$ are open subsets of S^n: For example

$$\widetilde{U}_j^+ = \{x \in \mathbb{R}^{n+1} \mid x^j > 0\} = \{x \in \mathbb{R}^{n+1} \mid x^j > 0\}$$

is open in $\mathbb{R}^{n+1}$ as the preimage of the open set $(0, \infty)$ under the continuous map $\mathbb{R}^{n+1} \to \mathbb{R}$ sending $x \mapsto x^j$. As we have that $U_j^+ = \widetilde{U}_j^+ \cap S^2$ the set U_j^+ is open in S^n in the subspace topology on S^n induced by $\mathbb{R}^{n+1}$. Similarly for U_j^-.

We claim that the maps $\psi_j^\pm : U_j^\pm \to \mathbb{R}^n$ sending

$$x = (x^0, \dots, x^n) \to (x^0, \dots, \widehat{x^j}, \dots, x^n)$$

define an atlas on S^n.

The image $\psi_j^\pm(U_j^\pm) = B^n \subset \mathbb{R}^n$ is the open unit ball for all j, in particular an open set. The map $\psi_j^\pm$ is continuous since preimages of small balls in B^n get mapped to small balls on the hemisphere which are open in the subspace topology of $S^n \subset \mathbb{R}^n$. Let $f : B^n \to \mathbb{R}$ be the smooth map $y \mapsto \sqrt{1 - y^2}$. Then for $x \in U_j^\pm$ by definition of the sphere:

$$x^j = \pm f(x^0, \dots, \widehat{x^j}, \dots, x^n).$$

(Hence $U_j^\pm$ is the graph of the map $\pm f$ after appropriately switching the coordinates $x^0, \dots, x^n$. Let us give the rest of the argument here not relying on the fact seen in the lecture that being locally a graph is enough to have an atlas.)

The inverse of $\psi_j^{\pm}$ (when we restrict the target space of the map $\psi_j^{\pm}$ to B^n) is the map $B^n \to U_j^{\pm}$ given by

$$y = (y^1, \dots, y^n) \mapsto (y^1, y^{j-1}, \pm f(y), y^j, \dots, y^n).$$

This map is continuous, in particular, $\psi_j^{\pm} : U_j^{\pm} \to B^n$ is a homeomorphism. This concludes the argument that $(\psi_j^{\pm}, U_j)$ is a system of charts on S^n.

Let us now prove that this system of charts is an atlas. First, the sets $U_j^{\pm}$ cover S^n (otherwise $x^j = 0$ for all j but this cannot happen for a point on the sphere). Second, we need to prove that all transition maps are smooth. Let $\varepsilon, \delta \in \{\pm 1\}$ be variables for the signs. Then overlaps look like

$$U_j^{\varepsilon} \cap U_k^{\delta} = \{x \in S^n \mid \varepsilon x^j > 0, \delta x^k > 0\}.$$

In particular, the overlap is empty for ($k = j$ and $\varepsilon = -\delta$), so nothing to prove in this case. For $j < k$, the transition functions are

$$\psi_j^{\varepsilon} \circ (\psi_k^{\delta})^{-1} : \{y \in B^n \mid \varepsilon y^j > 0\} \to U_j^{\varepsilon} \cap U_k^{\delta} \to \{y \in B^n \mid \delta y^k > 0\}$$

given by

$$(y^1 \dots, y^n) \mapsto (y^1, \dots, y^{k-1}, \delta f(y), y^k, \dots, y^n)$$
$$\mapsto (y^1, \dots, \widehat{y^j} \dots, y^{k-1}, \delta f(y), y^k, \dots, y^n)$$

which is smooth.

(b) Define $U^{\pm} = S^n \setminus \{N_{\pm}\}$ where $N_{\pm} = (0, \dots, 0, \pm 1)$ is the north pole pole N_+ and the south pole N_- with respect to the last coordinate x^n. Clearly, the two charts $U^{\pm}$ cover S^n. The stereographic projections $\psi_{\pm} : U^{\pm} \to \mathbb{R}^n$ are given by continuous functions

$$(x^0, \dots, x^n) \mapsto \frac{(x^0, \dots, x^{n-1})}{1 \mp x^n}.$$

It has inverse $\psi_{\pm}^{-1} : \mathbb{R}^n \to U^{\pm}$ given by

$$(y_1, \dots, y_n) \mapsto \frac{(2y_1, \dots, 2y_n, \pm(|y|^2 - 1))}{|y|^2 + 1}.$$

The overlap of the two charts is $U^+ \cap U^- = S^n \setminus \{N_+, N_-\}$. The transition function $\psi_- \circ \psi_+^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ is given by

$$(y_1, \dots, y_n) \mapsto \frac{y}{|y|^2}$$

which is smooth. Similarly for $\psi_+ \circ \psi_-^{-1}$. So the charts are compatible and we have an atlas on S^2.

2. An atlas on the real projective space

Let $\mathbb{RP}^n := \{\text{lines through the origin in } \mathbb{R}^{n+1}\}$. For $0 \neq x \in \mathbb{R}^{n+1}$, let $L = [x]$ be the line through x and 0.

(a) Define a suitable topology on $\mathbb{RP}^n$. (Hint: Define a metric on $\mathbb{RP}^n$.)

(b) For any $j \in \{0, \dots n\}$ define

$$\begin{aligned} \mathbb{R}^n_j &:= \{x = (x^0, \dots, x^n) \in \mathbb{R}^{n+1} \mid x^j = 0\} \cong \mathbb{R}^n, \\ Z_j &:= \{L \in \mathbb{RP}^n \mid L \subset \mathbb{R}^n_j\}, \\ U_j &:= \mathbb{RP}^n \setminus Z_j. \end{aligned}$$

Show that $U_j = \{[x] \in \mathbb{RP}^n \mid x^j \neq 0\}$ and that U_j is open in $\mathbb{RP}^n$.

(c) Define *homogeneous coordinates* on U_j by $\psi_j : U_j \to \mathbb{R}^n$ by[1]

$$\psi_j([x]) = \frac{(x^0, \dots, \widehat{x^j}, \dots x^n)}{x^j}.$$

Prove that the maps ψ_j are well-defined and that the coordinate systems (U_j, ψ_j) give an atlas on $\mathbb{RP}^n$.

Solution:

(a) Let $L, K \in \mathbb{RP}^n$ be two lines through the origin in $\mathbb{R}^{n+1}$. Then define

$$d_{\mathbb{RP}^n}(L, K) = \text{angle}(L, K) \in \left[0, \frac{\pi}{2}\right].$$

If the lines are $L = [x]$ and $K = [y]$, and θ the angle between L and K we have

$$\cos\theta = \frac{|\langle x, y\rangle_{\mathbb{R}^{n+1}}|}{|x||y|} \in [0, 1].$$

Note that the formula works for any choice of $x \in L$ and $y \in K$, as for another point $\tilde{x} = \lambda x \in L$ with $0 \neq \lambda \in \mathbb{R}$ the λ cancels out in the formula. Similarly for K. From another point of view: Suppose $x_\pm \in L$ and $y_\pm \in K$ are the intersection of the lines with the unit sphere $S^n \subset \mathbb{R}^{n+1}$. Then

$$d_{\mathbb{RP}^n}(L, K) = \min\{d^2_S(x_-, y_-), d^2_S(x_-, y_+)\}.$$

[1] The hat is a useful notation that means that this variable is omitted, i.e.

$$(x^0, \dots, \widehat{x^j}, \dots, x^n) := (x^0, \dots, x^{j-1}, x^{j+1}, \dots, x^n).$$

So in particular, if the distance (i.e. the angle) between two points $x, y \in S^n$ is smaller than $\frac{\pi}{2}$ then $d_{S^n}(x,y) = d_{\mathbb{RP}^n}([x],[y])$. (So S^n and $\mathbb{RP}^n$ are actually locally isometric).

(b) The lines $L = [x]$ in Z_j have are represented by points with $x^j = 0$, i.e. are orthogonal to the line $L_j = j$-axis, or equivalently, $L_j = [(0,\dots,0,1,0,\dots,0)]$ where the 1 is in the j-th entry. So lines in U_j are represented by points with $x^j \neq 0$, so have an angle strictly smaller than $\frac{\pi}{2}$. In the metric given above, U_j is the open ball of radius $\frac{\pi}{2}$ with center L_j.

(c) Suppose $L = [x] = [\tilde{x}]$. Then $\tilde{x} = \lambda x \in L$ with $0 \neq \lambda \in \mathbb{R}$. But

$$\psi_j([\tilde{x}]) = \frac{(\lambda x^0, \dots, \widehat{\lambda x^j}, \dots, \lambda x^n)}{\lambda x^j} = \frac{(x^0, \dots, \widehat{x^j}, \dots, x^n)}{x^j} = \psi_j([x]).$$

Hence ψ_j is well-defined.

The map U_j is a bijection: As $x^j \neq 0$ for $L = [x] \in U_j$, every line can uniquely be represented by some x of the form $(x^0, \dots, x^{j-1}, 1, x^{j+1}, \dots, x^n)$. So ψ_j is invertible with inverse $\psi_j^{-1} : \mathbb{R}^n \to U_j$ given by

$$(y_1, \dots, y_n) \mapsto (y^1, \dots, y^{j-1}, 1, y^j, \dots, y^n).$$

Moreover, ψ_j is a homeomorphism: As $x^j \neq 0$ for $L = [x] \in U_j$, every line can uniquely be represented by some $x \in S^n$ with $x^j > 0$, i.e. in the upper hemisphere of S^n with respect to the coordinate j. The map

$$x \mapsto \frac{(x^0, \dots, \widehat{x^j}, \dots x^n)}{x^j}$$

from $U_j^+ \subset S^n$ to $\mathbb{R}^n$ for U_j^+ as defined in exercise 1(a) is clearly a local homeomorphism. As S^n and $\mathbb{RP}^n$ are locally homeomorphic (as established in (a)) the map $\psi_j : U_j \to \mathbb{R}^n$ is a local homeomorphism. As the map ψ_j is also bijective it is a (global) homeomorphism.

Let us now prove that this system of charts is an atlas. First, the sets U_j cover $\mathbb{RP}^n$ (otherwise $x^j = 0$ for all j but this cannot happen as $x \neq 0$). Second, we need to prove that all transition maps are smooth. Then overlaps look like

$$U_j \cap U_k = \{[x] \in \mathbb{RP}^n \mid x^j \neq 0, x^k \neq 0\}.$$

Let us check now that the charts are compatible. For $j < k$, the transition functions are

$$\psi_j \circ (\psi_k)^{-1} : \{y \in \mathbb{R}^n \mid y^j \neq 0\} \to U_j \cap U_k \to \{y \in \mathbb{R}^n \mid y^k \neq 0\}$$

given by

$$(y^1 \dots, y^n) \mapsto (y^1, \dots, y^{k-1}, 1, y^k, \dots, y^n) \mapsto \frac{(y^1, \dots, \widehat{y^j}, \dots, y^{k-1}, 1, y^k, \dots, y^n)}{y^j}$$

which is smooth.

3. Two diffeomorphic but not equal structures on the real line

Consider $\mathbb{R}$ with its usual differentiable structure, induced by the chart $\varphi : \mathbb{R} \to \mathbb{R}$, $\varphi(x) = x$. Also consider the differentiable structure induced by the chart $\psi : \mathbb{R} \to \mathbb{R}$, $\psi(x) = x^3$.

Show that the two differentiable structures are not equal, but that nevertheless, the two differentiable manifolds are diffeomorphic.

Solution:

Both are global charts. The transition function $\psi \circ \varphi^{-1} : \mathbb{R} \to \mathbb{R}$ is given by $x \mapsto x^3$ which is smooth, bijective with continuous but not smooth inverse, so the two charts are not compatible. However, to see that the both structures on $\mathbb{R}$ are diffeomorphic, we need to find a map $f : \mathbb{R} \to \mathbb{R}$ which is a diffeomorphism when we equip the first $\mathbb{R}$ with the atlas induced by φ and we equip the second $\mathbb{R}$ with the atlas induced by ψ. More concretely, we want to have an f such that the composition from left to right in

$$\mathbb{R} \xleftarrow{\varphi} \mathbb{R} \xrightarrow{f} \mathbb{R} \xrightarrow{\psi} \mathbb{R}$$

is a diffeomorphism. Set $f(x) = \psi^{-1}(x)$ then the above composition is the identity.

4. Quaternions

Let Q denote the vector space $\mathbb{R}^4$ with basis $\{1, i, j, k\}$ and multiplication subject to the laws $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. (These make Q into an *algebra*.)

(a) Show that every non–zero element $u \in Q$ is invertible.

Hint: Set $u = a + bi + cj + dk$. It is useful to define the *conjugate* $\bar{u} := a - bi - cj - dk$ and to prove $\bar{u}u = |u|^2 = u\bar{u}$.

(b) Show that $|uv| = |u||v|$ for $u, v \in Q$.

(c) Show that $S^3 := \{u \in Q \mid |u| = 1\}$ has the structure of a group.

Solution:

(a) Let us prove the hint:

$$\begin{aligned}\bar{u}u &= (a{+}bi + cj + dk)(a - bi - cj - dk)\\ &= a^2{-}abi - acj - adk + abi - bi^2 - bcij - bdik\\ &\qquad + acj - bcji - c^2j^2 - cdjk + adk - bdki - cdkj - d^2k^2\\ &= a^2{+}b^2 + c^2 + c^2 = |u|^2\end{aligned}$$

Similarly, $u\bar{u} = |u|^2$. We see immediately that $v = \frac{\bar{u}}{|u|^2}$ is an inverse for a given non-zero $u \in Q$.

(b) **Direct proof:**

Let $u = a + bi + cj + dk$ and $v = e + fi + gj + hk$. Then $|uv|^2$ is equal to

$$\begin{aligned}&|(a{+}bi + cj + dk)(e + fi + gj + hk)|^2 =\\ &= |ae{+}afi + agj + ahk + bei - bf + bgk - bhj\\ &\qquad + cej - cfk - cg + chi + dek + dfj - dgi - dh|^2\\ &= |ae - bf - cg - dh + (af + be + ch - dg)i\\ &\qquad + (ag - bh + ce + df)j + (ah + bg - cf + de)k|^2\\ &= (ae - bf - cg - dh)^2 + (af + be + ch - dg)^2\\ &\qquad + (ag - bh + ce + df)^2 + (ah + bg - cf + de)^2\\ &\qquad = (a^2 + b^2 + c^2 + d^2)(e^2 + f^2 + g^2 + h^2)\end{aligned}$$

which is $|u|^2|v|^2$.

Alternative proof:

Show that $\overline{uv} = \bar{v}\bar{u}$ by direct computation. Then we get

$$|uv|^2 = uv\overline{uv} = uv\bar{v}\bar{u} = u|v|^2\bar{u} = u\bar{u}|v|^2 = |u|^2|v|^2.$$

(c) The identity element is 1. As multiplication preserves the norm by (b) we have $uv \in S^3$ for $u, v \in S^3$. Moreover, if $u \in S^3$ then its inverse is $\bar{u}$ which is also in S^3.

5. For those new to topology

(a) Prove that the subspace topology is a topology.

(b) Prove that the quotient topology is a topology.

(c) Show that the subspace topology for S^1 in $\mathbb{R}^2$ coincides with the quotient topology $\mathbb{R} \to S^1$.

(d) Recall that a topological space X is not connected if there are non-empty, open, disjoint subsets $U, V \subset X$ such that $U \cup V = X$. Any U that arises this way (and X itself) is called a component of X. Show that if X is a manifold there is a unique decomposition

$$X = \bigcup_{\alpha} U_\alpha,$$

where U_α are disjoint connected components of X.

(e) What happens if we want to decompose the Cantor set X as in (d)?

Solution:

(a) Let (X, τ) be a topological space and $A \subset X$ a subset. Then the subspace topology on A induced by (X, A) is given by

$$\tau_A = \{U \cap A \,|\, U \in \tau\}.$$

To check that this defines a topology, let us check the axioms:

- $\emptyset = \emptyset \cap A \in \tau_A$ and $A = X \cap A \in \tau_A$ as $\emptyset, X \in \tau$.
- Suppose $\tilde{U}, \tilde{V} \in \tau_A$, i.e. there are $U, V \in \tau$ such that $\tilde{U} = U \cap A$ and $\tilde{V} = V \cap A$. Then $\tilde{U} \cap \tilde{V} = (U \cap V) \cap A \in \tau_A$ as $U \cap V \in \tau$ using that τ is a topology and thus $U \cap V \in \tau$.
- Suppose $\tilde{U}_\beta \in \tau_A$ for all β in some index set B, then for each $\beta \in B$ there is $U_\beta \in \tau_A$ such that $\tilde{U}_\beta = u_\beta \cap A$. Then

$$\bigcup \tilde{U}_\beta = \bigcup (U_\beta \cap A) = \Big(\bigcup U_\beta\Big) \cap A \in \tau_A$$

  using that τ is a topology and thus $\bigcup U_\beta \in \tau$.

(b) Let (X, τ) be a topological space and $\sim$ an equivalence relation on X. Denote $X/_\sim$ the space of equivalence classes and $\pi : X \to X/_\sim$ given by $p \mapsto [p]$ the canonical surjection, sending p to its equivalence class. Then the quotient topology on $X/_\sim$ is given by

$$\tau_\sim = \{U \subset X/_\sim \,|\, \pi^{-1}(U) \in \tau\}.$$

Note that by definition preimages of open sets are open, so π is continuous. To check that this defines a topology, let us check again the axioms:

- $\pi^{-1}(\emptyset) = \emptyset \in \tau$ and $\pi^{-1}(X/_\sim) = X \in \tau$. Hence $\emptyset, X \in \tau_\sim$.
- Suppose $U, V \in \tau_\sim$, i.e. $\pi^{-1}(U), \pi^{-1}(V) \in \tau$. Then $\pi^{-1}(U \cap V) = \pi^{-1}(U) \cap \pi^{-1}(V) \in \tau$ as τ is a topology and hence unions of elements in τ are again in τ.

- Suppose $U_\beta \in \tau_\sim$ for all β in some index set B, then for each $\beta \in B$ $\pi^{-1}(U_\beta) \in \tau$ by definition of the subspace topology. But then

$$\pi^{-1}\left(\bigcup U_\beta\right) = \bigcup \left(\pi^{-1} U_\beta\right) \in \tau$$

using that τ is a topology and infinite unions of elements in τ are again in τ.

(c) Let τ_S be the subspace topology on S^1 coming from $S^1 \subset \mathbb{R}^2$ and τ_Q be the quotient topology coming from $\mathbb{R} \to \mathbb{R}/_{x \sim x+2\pi x}$. Let $\tilde{U} \in \tau_S$, i.e there is a $U \in \tau_{\mathbb{R}^2}$ such that $U \cap S^1 = \tilde{U}$. So for any $p \in \tilde{U}$ there is an open disk $D \subset U$ with p as the center. Note that $D \cap S^1$ is open in S^1. For D small enough $D \cap S^1$ is just an open circular arc I. If we look at S^1 as a quotient of $\mathbb{R}$ then the circular segment corresponds to an open interval $\tilde{I}$. This is open in the quotient topology τ_Q as $\pi^{-1}(\tilde{I}) = I + 2\pi\mathbb{Z}$ is open in $\mathbb{R}$ for I an interval that projects to $\tilde{I}$. The converse follows by the argument backwards. For a point in an open set in the quotient topology, there is an interval $\tilde{I}$, which corresponds to a circular arc in $S^1 \subset \mathbb{R}^2$.

(d) First note that two distinct connected components must be disjoint, because if U, V are connected components and the intersection $U \cap V$ is nonempty, then $U \cup V$ is open and connected ($U \cup V$ not connected would imply that either U or V is not connected by using a disjoint nontrivial decomposition of $U \cup V$ into disjoint sets.).

Let now $x \in X$ we need to show that x is contained in some connected component U_x. For that let

$$\mathcal{U} = \{U \subset X \,|\, U \text{ open, connected and} x \in U\}.$$

Then $U_x = \bigcup U \in \mathcal{U} U$. Here we used that X is a manifold: $\mathcal{U}$ is not empty, as for any point x in X there is an open set $V \subset X$ containing x which is homeomorphic by a map ψ to a nonempty open subset $\tilde{V} \subset \mathbb{R}^n$. But taking the preimage of an open ball $B \subset \tilde{V}$ that contains $\psi(x)$, produces a connected open subset $\psi^{-1}(B)$ which contains x. Actually, the claimed decomposition statement is true for any *locally connected space* and manifolds are locally connected.

(e) The Cantor set is not locally connected. Any nonempty open subset of the Cantor set C is disconnected, hence C cannot be the union of connected open subsets.