Exercise Sheet 6

To be handed in until November 01

1. A smooth projection Prove: The projection $\pi : S^n \to \mathbb{RP}^n$, $x \mapsto [x]$ is smooth.

Solution:

Let us recall the charts from exercises 1 and 2 from sheet 5. Recall that

$$S^n \supset U_j^{\pm} = \{x \in S^n \mid \pm x^j > 0\} \xrightarrow{\psi_j^{\pm}} \mathbb{R}^n$$

and

$$\mathbb{RP}^n \supset U_j = \{x \in S^n \,|\, x^j \neq 0\} \xrightarrow{\psi_j} B^n$$

are charts for $j = 0, \ldots, n$ that define an atlas on S^n and on $\mathbb{RP}^n$, respectively. Note that $\pi|_{U_j^{\pm}} : U_j^{\pm} \to U_j$ is a bijection for every j, actually a homeomorphism.

Therefore, to prove that π is smooth it is enough to show that the composition

$$B_n = \psi_j^{\pm}(U_j^{\pm}) \xrightarrow{(\psi_j^{\pm})^{-1}} U_j^{\pm} \xrightarrow{\pi} U_j \xrightarrow{\psi_j} \psi_j(U_j) = \mathbb{R}^n$$

is smooth. The composition is given by

$$y \mapsto \frac{y}{\pm\sqrt{1 - y_j^2}}$$

which is smooth for all j. This proves that π is smooth.

2. The orthogonal group Let $O(n) := \left\{A \in \mathbb{R}^{n\times n} \mid A^T A = I\right\}$ be the group of orthogonal matrices. Characterize the tangent space $T_I O(n)$ of $O(n)$ at the identity I as follows.

(a) Let $A(t)$ be a smooth curve in $\mathbb{R}^{n\times n}$ with $A(0) = I$, $A(t) \in O(n)$. Find an equation satisfied by $B := dA(0)/dt$.

(b) The exponential map for sqaure matrices $C \in \mathbb{R}^{n\times n}$ is given by

$$e^C = \sum_{k=0}^{\infty} \frac{C^k}{k!}.$$

Prove that if $A(t) = e^{Bt}$ then $\frac{d}{dt}A(t) = Be^{Bt}$.

(c) For any B satisfying the equation from a) find a curve $A(t)$ in $O(n)$ with initial velocity B.

(d) A very beautiful picture of any such $A(t)$ comes by considering the diagonalization of orthogonal matrices to 2×2 blocks. Write $\mathbb{R}^n$ as the orthogonal sum of 2 dimensional subspaces V_i (and possible a 1-dimensional subspace W) and set each V_i rotating at constant angular speed θ_i.

(e) What is the dimension of $O(n)$?

Solution:

(a) Let $A(t)$ be a smooth curve in $\mathbb{R}^{n\times n}$ with $A(0) = I$, $A(t) \in O(n)$ for $t \in (-\varepsilon, \varepsilon)$. The latter property is

$$A(t)^T A(t) = I$$

for all t. Taking the derivative of this equation yields

$$\left(\frac{d}{dt}A(t)\right)^T A(t) + A(t)^T \left(\frac{d}{dt}A(t)\right) = 0$$

So for $t = 0$, using that $A(0) = I$ we get

$$\left(\frac{d}{dt}A(0)\right)^T + \left(\frac{d}{dt}A(0)\right) = 0.$$

So $B := \frac{d}{dt}A(0)$ is an antisymmetric matrix.

(b) We have $\frac{d}{dt}A(t) = Be^{Bt} = e^{Bt}B$ using componentwise derivative.

(c) Suppose B is an anti-symmetric matrix. Then define $A(t) = e^{tB}$. By b) $\frac{d}{dt}A(0) = Be^0 = B$.

We are left to show that $A(t)$ is in $O(n)$ for any t. Recall that if C and D are matrices that commute then $e^{C+D} = e^C e^D$. But tB and tB^T commute as

$$BB^T = B(-B) = -B^2 = (-B)B = B^T B$$

using that antisymmetry $B^T = -B$. Hence

$$A(t)^T A(t) = \left(e^{tB}\right)^T = e^{tB} = e^{tB^T} e^{tB} = e^{tB^T + tB} = e^{t(B^T+B)} = e^0 = I.$$

Let us give a second proof of the fact that $A(t)$ is in $O(n)$. Using b)

$$\begin{aligned}
\frac{d}{dt}(A(t)^T A(t)) &= \left(\frac{d}{dt}A(t)\right)^T A(t) + A(t)^T \left(\frac{d}{dt}A(t)\right) \\
&= B^T e^{tB^T} e^{tB} + e^{tB^T} B e^{tB} \\
&= (B^T + B)e^{tB^T} e^{tB} = 0.
\end{aligned}$$

So $A(t)^T A(t)$ is constant and as $A(0)^T A(0) = I^T I = I$ we have $A(t)^T A(t)$ for all t.

(d) Antisymmetric 2×2-matrices are of the form

$$B_\theta = \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix}$$

for $a \in \mathbb{R}$. They are a one-dimensional linear subspace of all 2×2-matrices. The corresponding curve in $O(2)$ is

$$A_\theta(t) = e^{tB_\theta} = \begin{pmatrix} \cos(\theta t) & -\sin(\theta t) \\ \sin(\theta t) & \cos(\theta t) \end{pmatrix}.$$

For a general antisymmetric $n \times n$ matrix B let $A = e^B$. Then by linear algebra, for n even there are angles $\theta_1, \dots, \theta_{n/2}$ and such that with respect to a base change to 2-dimensional subspaces $V_1, \dots, V_{n/2}$ the matrix A is

$$A = \operatorname{diag}(R_{\theta_1}, \dots, R_{\theta_{n/2}}).$$

where $R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$. But then

$$A(t) = e^{tB} = \operatorname{diag}(R_{\theta_1 t}, \dots, R_{\theta_{n/2} t})$$

in the basis given by the subspaces V_j. Similarly for n odd

$$A(t) = e^{tB} = \operatorname{diag}(R_{\theta_1 t}, \dots, R_{\theta_{(n-1)/2} t}, \pm 1).$$

(e) Let E_{jk} be the matrix having everywhere entry 0 except being 1 in row j and column k. The linear subspace of antisymmetric matrices is of vector space dimension $\frac{n(n-1)}{2}$ with basis given by $E_{jk} - E_{jk}$ for $j < k$. There are exactly $\binom{n}{2} = \frac{n(n-1)}{2}$ pairs (j, k) with $j < k$. The dimension of the tangent space is the same as the dimension of the manifold, hence

$$\dim O(n) = \frac{n(n-1)}{2}.$$

3. Cutoff functions Let M be a smooth manifold. For a function $u : M \to \mathbb{R}$ define the *support of* u as

$$\operatorname{spt}(u) := \overline{\{p \in M \mid u \neq 0\}}.$$

We say that a set Z is *compactly contained* in an open set $U \subset M$ if $\overline{Z}$ is compact and $\overline{Z} \subset U$. Write $K \subset\subset U$ in this case. Prove: if U is an open set in M and K a compact subset of U, then there exists a *cutoff function for* K *in* U, i.e. a function $\chi : M \to \mathbb{R}$ such that

i) χ is smooth,

ii) $0 \leq \chi \leq 1$,

iii) $\chi \equiv 1$ on K,

iv) $\operatorname{spt}(\chi) \subset\subset U$.

Hint: Recall without proof from analysis that the function

$$f(x) := \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0, \end{cases}$$

is a smooth map.

Solution:

Claim. There is a smooth function $f_1 : \mathbb{R} \to [0,1]$ such that f_1 is 0 on $(-\infty, 0]$ and is 1 on $[1, \infty)$.

Proof. Set

$$f_1 = \frac{f(x)}{f(x) + f(1-x)}.$$

□

Claim. There is a smooth function $f_2 : \mathbb{R}^n \to [0,1]$ such that f_2 is 0 on $|x| \geq 2$ and is 1 on $|x| \leq 1$.

Proof. Set

$$f_2 = f_1(1 - |x|).$$

□

Claim. Let $\tilde{K} \subset \mathbb{R}^n$ be compact. Then there is a smooth compactly supported function $f_{\tilde{K}} : \mathbb{R}^n \to [0,1]$ which is 1 on $\tilde{K}$.

Proof. As $\tilde{K}$ is compact it is bounded, so $\tilde{K} \subset B_R(0)$ for some $R > 0$. Then set $f_{\tilde{K}}(x) = f_2(x/R)$. □

Now to the manifold M and the sets $K \subset U \subset M$. As $U \subset M$ is open, U is a manifold itself. Only look at charts of the manifold U that are homeomorphisms $\psi_j : U_j \to \mathbb{R}^n$ for $U_j \subset U$ open, i.e $\psi_j(U_j) = \mathbb{R}^n$. As K is compact we can cover K by finitely many such charts $U_1, \ldots, U_k$ with homeomorphisms $U_j \to \mathbb{R}^n$ and such that $\psi_1^{-1}(B_1(0)), \ldots, \psi_k^{-1}(B_1(0))$ cover K. Note that

$$K = \bigcup_{j=1}^{k} K_j$$

is a union of compact sets $K_j := K \cap \psi_j^{-1}(\overline{B_1(0)}) \subset K$. The sets $\tilde{K}_j := \psi_j(K_j) \subset \mathbb{R}^n$ are compact, so there are compactly supported functions $\tilde{g}_j : \mathbb{R}^n \to [0,1]$ such that $\tilde{g}_j$ is 1 on $\tilde{K}_j$. The function

$$g_j(x) = \begin{cases} \tilde{g}_j(\psi_j(x)) & x \in U_j, \\ 0 & x \in M \end{cases}$$

is well-defined and smooth as $\tilde{g}_j$ is compactly supported. Moreover, the support of g_j is the preimage of the support of $\tilde{g}_j$ under ψ_j. Hence g_j is compactly supported with support contained in U.

Next define the smooth function $g : M \to [0, \infty)$ by

$$g(x) = \sum_{j=1}^{k} g_j(x).$$

Note that because every $x \in K$ is contained in some K_j that $g_j(x) = 1$ and hence $g(x) \geq 1$. The function $\chi : M \to [0,1]$ defined by $\chi(x) = f_1(g(x))$ satisfies properties i)-iv).

4. Coordinate vector fields are linearly independent Prove that

$$\left(\frac{\partial}{\partial x^1}\right)_{p,\psi}, \ldots, \left(\frac{\partial}{\partial x^n}\right)_{p,\psi}$$

are linearly independent tangent vectors.

Hint: Compute

$$\left(\frac{\partial}{\partial x^j}\right)_{p,\psi} \cdot (\zeta x^k),$$

where $x^1, \ldots, x^n : U \to \mathbb{R}$ are the coordinate functions associated to (U, ψ) and ζ is a cutoff function for p in U (i.e. equal to 1 in a neighbourhood of p.)

Solution:

Recall from the lecture that for a smooth function $M \to \mathbb{R}$ we have the formula

$$\left(\frac{\partial}{\partial x^j}\right)_{p,\psi} f = \frac{\partial \tilde{f}}{\partial x^j}(\tilde{p})$$

for $\tilde{f} = f \circ \psi^{-1} : \psi(U) \to \mathbb{R}$ and $\tilde{p} = \psi(p)$. The function $f_k = \zeta x^k$ is a well-defined smooth map $M \to \mathbb{R}$. Note that $\tilde{f}_k : \psi(U) \to \mathbb{R}$ is by definition the coordinate map $\mathbb{R}^n \ni x \to x^k \in \mathbb{R}$ near $\tilde{p} = \psi(p)$. So

$$\left(\frac{\partial}{\partial x^j}\right)_{p,\psi} (\zeta x^k) = \frac{\partial x^j}{\partial x^k}(\psi(\tilde{p})) = \delta_{jk} := \begin{cases} 1 & j = k, \\ 0 & j \neq k. \end{cases}.$$

This proves that the maps

$$\left(\frac{\partial}{\partial x^1}\right)_{p,\psi}, \ldots, \left(\frac{\partial}{\partial x^n}\right)_{p,\psi} : C^\infty(M) \to \mathbb{R}$$

are linearly independent as elements of of the vector space $\mathrm{Hom}(C^\infty(M), \mathbb{R})$.