Exercise Sheet 7

To be handed in until November 08

1. The Veronese embedding

(a) Consider the map $F : \mathbb{R}^3 \to \mathbb{R}^4$ given by $F(x, y, z) := (x^2 - y^2, xy, xz, yz)$. Prove that F induces a well-defined map $f : \mathbb{RP}^2 \to \mathbb{R}^4$ characterized by $f([p]) := F(p)$ for any $p \in S^2$.

(b) Prove that f is injective.

(c) Prove that f is an immersion.

(d) Prove that f is a homeomorphism onto its image. (The map f is called the *Veronese embedding* of $\mathbb{RP}^2$ in $\mathbb{R}^4$. Note that $\mathbb{RP}^2$ does not embed in $\mathbb{R}^3$.)

Solution:

(a) F is given by homogeneous polynomials of even degree. That is $F(\lambda x) = \lambda^2 F(x)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$. More concretely, for $p \in S^2$ we see that $F(-p) = (-1)^2 F(-p) = F(-p)$ as the minus cancels everywhere. Hence $f([p]) = f([-p])$ so f is well-defined.

(b) Suppose $f([p]) = f([q])$ for $p = (x, y, z), q = (u, v, w)$ with $|p| = |q| = 1$. We want to show that $p = \lambda q$ for some $\lambda \in \{\pm 1\}$. By assumption, we have the following equations

$$\begin{aligned} x^2 - y^2 &= u^2 - v^2, \\ xy &= uv, \\ xz &= uw, \\ yz &= vw. \end{aligned}$$

If $x = 0$ then either u or v is 0 by the second equation. Suppose $u = 0$. Then by the first equation $y = \pm v$. If now $y = \pm v \neq 0$ it follows by the last equation that $z = \pm w$ and hence indeed $p = q$ or $p = -q$. If $y = v = x = w = 0$ then as p, q are on the sphere, they are either $(0, 0, 1)$ or $(0, 0, -1)$ and hence also equal or antipodal.

The same argument works when $x = 0$ and $v = 0$. If $y = 0$, we again get either that u or v is 0.

So assume $x \neq 0$ and $y \neq 0$. Then

$$z^2 = \frac{uw}{x}\frac{vw}{y} = \frac{uvw^2}{xy} = w^2$$

So $z = \pm w$. If $z = \pm w$ are non-zero, we get $y = \pm v$ and then also $x = \pm u$ by the middle two equations, so $p = \pm q$. If $z = w = 0$ then as p, q lie on the sphere $x^2 + y^2 = 1 = u^2 + v^2$, so adding this to the first equation we get $2x^2 = 2u^2$, so again $x = \pm u$. Then also $y = \pm v$ and consequently $p = \pm q$.

(c) To prove that f is smooth we need to prove that $f \circ \psi_j^{-1} : \mathbb{R}^2 \to \mathbb{R}^4$ is smooth where $\psi_j : U_j \to \mathbb{R}^2$ are the chart from exercise sheet 5 exercise 2 for $j = 0, 1, 2$. The maps are

$$(u,v) \overset{\psi_j^{-1}}{\mapsto} \begin{cases} [1,u,v] \overset{f}{\mapsto} (1-v^2, u, v, uv)/r^2, & j = 0, \\ [u,1,v] \mapsto (u^2-1, u, uv, v)/r^2, & j = 1, \\ [u,v,1] \mapsto (u^2-v^2, uv, u, v)/r^2, & j = 2. \end{cases}$$

which are all smooth. Here $r^2 = 1^2 + u^2 + v^2$.

The derivatives of $f \circ \psi_j^{-1}$ at $(u,v) = \psi_j([p])$ are the linear maps $\mathbb{R}^2 \to \mathbb{R}^4$ represented by the matrix

$$D(f \circ \psi_1^{-1})(u,v) = \frac{1}{r^4}\begin{pmatrix} -2u(1-v^2) & -2vr^2 - 2v(1-v^2) \\ r^2 - 2u^2 & -2uv \\ -2uv & r^2 - 2v^2 \\ vr^2 - 2u^2v & ur^2 - 2uv^2 \end{pmatrix},$$

$$D(f \circ \psi_2^{-1})(u,v) = \frac{1}{r^4}\begin{pmatrix} 2ur^2 - 2u(u^2-1) & -2v(u^2-1) \\ r^2 - 2u^2 & -2uv \\ vr^2 - 2u^2v & ur^2 - 2uv^2 \\ -2uv & r^2 - 2v^2 \end{pmatrix},$$

$$D(f \circ \psi_3^{-1})(u,v) = \frac{1}{r^4}\begin{pmatrix} 2ur^2 - 2u(u^2-v^2) & -2vr^2 - 2v(u^2-v^2) \\ vr^2 - 2u^2v & ur^2 - 2uv^2 \\ r^2 - 2u^2 & -2uv \\ -2uv & r^2 - 2v^2 \end{pmatrix}.$$

The three maps are injective for all pairs $(u,v) \in \mathbb{R}^2$, hence also $D_{[p]}f : T_{[p]}M \to \mathbb{R}^4$ is injective for every $[p] \in \mathbb{RP}^2$. This proves that f is an immersion.

(d) Note that $\mathbb{RP}^n$ is compact for every n as the image of a compact set under the continuous map $S^n \to \mathbb{RP}^n$. From topology we know that a continuous injective map from a compact Hausdorff domain is automatically a homeomorphism onto its image.

2. TS^3 has a global trivialization

A *Lie group* is a smooth manifold endowed with a group structure such that the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth.

(a) Show that $S^3 := \{V \in Q : |V| = 1\}$ (where Q are the quaternions) is a Lie group.

(b) Construct smooth vector fields X, Y, Z on S^3 such that $X(u)$, $Y(u)$, $Z(u)$ are independent for each u in S^3. Conclude that $TS^3 \cong S^3 \times \mathbb{R}^3$.

Solution:

(a) Let $g \in U_1, h \in U_2$ and $gh \in U_3$, where $\psi_{j_r} : U_{j_r} \to B^n$ are hemisphere charts as described in exercise 1 sheet 5 for $j_1, j_2, j_3 \in \{0, 1, 2, 3\}$. Denote by $m : S^3 \times S^3 \to S^3$ the multiplication in S^3. Then $\psi_{j_3} \circ m \circ (\psi_{j_1} \times \psi_{j_2})^{-1} : B^n \times B^n \to B^n$ is given by the composition of smooth functions as the product of quaternions $Q \times Q \to Q$ is smooth. Same for the inverse.

(b) Let $u \in S^3 \subset Q \cong \mathbb{R}^4$. Then $iu, ju, ku \in Q \cong \mathbb{R}^4$ span T_uS^3. Let us prove that iu is in the tangent space of S^3 at u. Identify $Q \cong \mathbb{C} \times \mathbb{C}$ by $u = z + jw$ as in exercise 4. Then $iu = iz + ijw$ is orthogonal to u as

$$\langle u, iu \rangle_{\mathbb{R}^4} = \langle z, iz \rangle_{\mathbb{R}^2} + \langle w, iw \rangle_{\mathbb{R}^2} = 0$$

as multiplication by i rotates a complex number by $\pi/2$ and so z is perpendicular to iz and w is perpendicular iw. Similarly show that

$$u, iu, ju, ku$$

are an orthonormal basis of $\mathbb{R}^4$.

Hence the vector fields X, Y, Z defined by $X(u) = iu, Y(u) = ju, Z(u) = ku$ are pointwise independent vector fields.

Whenever we find $X_1, \dots, X_n$ pointwise linearly independent vector fields on a n-dimensional manifold M the map $M \times \mathbb{R}^n \to TM$ given by

$$(p; a_1, \dots, a_n) \mapsto \left(p, \sum_{r=1}^{n} a_r X_r(p)) \right)$$

is an diffeomorphism.

As we found 3 vector fields on S^3 that are pointwise linearly independent, we have $S^3 \times \mathbb{R}^3 \cong TS^3$.

3. The Hopf fibration

(a) Prove that every sphere of odd dimension carries a nowhere vanishing vector field.

(b) Prove that S^{2n-1} has a "smooth" decomposition into circles. They are called *Hopf fibers* of S^{2n-1}.

(c*) Can S^2 be decomposed into a disjoint union of submanifolds diffeomorphic to S^1?

Solution:

(a) Look at $S^{2n-1} \subset \mathbb{R}^{2n} \cong \mathbb{C}^n$. Then define $X(p) = ip$. That is if $p = (z_1, \ldots, z_n) \in \mathbb{C}^n$ then $X(p) = ip = (iz_1, \ldots, iz_n)$. The vector $X(p)$ is orthogonal to p:

$$\langle p, X(p)\rangle_{\mathbb{R}^{2n}} = \langle p, ip\rangle_{\mathbb{R}^{2n}} = \sum_{j=1}^{n} \langle z_j, iz_j\rangle_{\mathbb{R}^2} = 0$$

as multiplication by i rotates a complex number by $\pi/2$ and so iz_j is perpendicular to the original complex number z_j. The orthogonality of p and $X(p)$ proves that $X(p)$ is a tangent vector at p. Since $|X(p)| = |ip| = |p| = 1$ for any p the vector field X defines a nowhere vanishing vector field on S^{2n-1}.

(b) For any $p \in S^{2n-1}$ look at the great circle $\gamma_p : \mathbb{R} \to S^{2n-1}$ given by $t \mapsto e^{it}p$. It parametrizes a circle as $\gamma(t + 2\pi) = \gamma(t)$. Its tangent vector is $\gamma_p'(t) = ie^{it}p = i\gamma_p(t) = X(\gamma_p(t))$. So γ follows the vector field X. This is the reason that two different such circles have either the same or a disjoint image. Indeed, suppose $q \in \gamma_p(t)$, i.e. $q = e^{it_0}p$ for some $t_0 \in \mathbb{R}$. Then $\gamma_q(t) = e^{it}q = e^{i(t+t_0)}p$, so in particular the image of γ_p and γ_q agree. Conversely, if $q \neq e^{it}p$ for any $t \in \mathbb{R}$ then also $qe^{is} \neq e^{it}p$ for any $t \in \mathbb{R}, s \in \mathbb{R}$. So the circles are disjoint. (You could also prove that being on the same circle is an equivalence relation and then use that equivalence classes form a partition of a set.)

(c) No. We need the smooth version of the Jordan-Schoenflies Theorem:

Theorem. (Smooth Jordan-Schoenflies Theorem)

(i) Deleting an embedded circle in S^2 divides S^2 into two parts that are diffeomorphic to open disks.

(ii) Deleting an embedded circle in an open disk D divides D into two parts, one diffeomorphic to an open disk and one diffeomorphic to an open annulus.

Suppose S^2 can be decomposed into a disjoint union of submanifolds diffeomorphic to S^1. Choose one circle C_0. By the smooth Jordan-Schoenflies Theorem $S^2 \setminus C_0$ is diffeomorphic to two disks. Choose one of these disks and call it D. Every circle must entirely lie in D or in $S^2 \setminus D$ as the circles are assumed to be disjoint. Again by the smooth Jordan-Schoenflies Theorem any circle C in D divides D into two components D_C and A_C, where D_C is diffeomorphic to an open disk and and A_C diffeomorphic to an open annulus. So for any other circle $\tilde{C}$ either $\tilde{C} \subset A_C$ or $\tilde{C} \subset D_C$. We can define a partial order on the set of circles that are contained in D. Define

$$C_1 < C_2 \qquad \text{iff} \qquad C_1 \subset D_{C_2}.$$

Note that if $C_1 < C_2$ then also $D_{C_1} \subset D_{C_2}$ and $\operatorname{area}(D_{C_1}) < \operatorname{area}(D_{C_2})$ as $D_{C_2} \setminus D_{C_1}$ is an open annulus, and open sets have always positive area as they contain a round disk. Let $a = \inf \operatorname{area}(D_C)$ for circles C in D. We claim that a is not just an infimum but the infimum is realized by the area of some disk $D_{C'}$ for some circle C' in D. Suppose $C_1 > C_2 > \ldots$ is a descending sequence such that $a = \inf \operatorname{area}(\overline{D_{C_n}})$. Since $C_1 > C_2 > \ldots$ also $D_{C_1} \supset D_{C_2} > \ldots$ and as the closures $\overline{D_{C_1}} \supset \overline{D_{C_2}} > \ldots$ are a descending sequence of compact sets the intersection

$$A = \bigcap_{n \in \mathbb{N}} \overline{D_{C_n}}$$

is non-empty. As it is non-empty there must be a point $x \in A$. This point x must lie in a circle C' which also bounds a disk $D_{C'}$. But as $x \in \overline{D_{C_n}}$ for all n and $\overline{D_{C_n}} \subset D_{C_{n-1}}$ we have $D_{C'} \subset D_{C_n}$ for all n. In particular, $\operatorname{area}(D_{C'}) \leq \operatorname{area}(D_{C_n})$ for all n implies by the definition of the sequence C_n infimizing the areas of D_{C_n} that $a = \operatorname{area}(D_{C'})$.

Note that $a > 0$ as open disks must have a positive area. On the other hand, if $a > 0$ then choose a point in $D_{C'}$. Then there is a circle going through this point that again bounds a disk of area strictly less than a which contradicts the definition of a. We get a contradiction, so S^2 cannot be decomposed into a disjoint union of circles (we actually also proved that $\mathbb{R}^2$ cannot be written as the disjoint union of circles).

4. Visualization of the Hopf fibration for S^3

Identify $\mathbb{C}^2$ with the quaternions Q by identifying $(z, w) = (a + bi, c + di) \in \mathbb{C}^2$ with $z + wj = a + bi + cj + dk \in Q$.

(a) Indentify $S^3 \setminus \{-1\}$ with $\mathbb{R}^3$ via stereographic projection from the point $-1 \in Q$. Locate in the target $\mathbb{R}^3$ the images of the points $1, \pm i, \pm j, \pm k$ and the 6 "coordinate" great circles of S^3.

(b) For $0 \leq r \leq \pi/2$, define

$$T_r := \{(z, w) : |z| = \cos(r), |w| = \sin(r)\}.$$

(i) Observe that $(T_r)_{0 \leq r \leq \pi/2}$ is a partition of S^3.

(ii) Observe that T_0 and $T_{\pi/2}$ are great circles of S^3 and are Hopf fibers in the sense of exercise 3.

(iii) Observe that for $0 < r < \pi/2$ the T_r are all tori, that they are equidistant from each other (in the path metric on S^3) and that each T_r is a union of Hopf fibers. The middle torus $T_{\pi/4} = \{(z, w) \,|\, |z| = |w| = 1/\sqrt{2})\}$ is called the *Clifford torus*.

(c) Visualize the Hopf fibration of S^3 by drawing all of the Hopf fibers in $\mathbb{R}^3$ (after stereographic projection). The tori T_r are useful guides.

(d) Remarkably, the quotient space $S^3/\sim$ is S^2, where $\sim$ is the equivalence relation where each Hopf fiber becomes a point. Can you "see" the S^2 that is swept out as the fiber S^1 varies in S^3? Can you find the upper and lower hemispheres of the S^2 in your diagram?

Solution:

(a) Stereographic projection with respect to $-1 = (-1, 0, 0, 0) \in \mathbb{R}^4$ is $\psi : S^3 \setminus \{-1\} \to \mathbb{R}^3$ given by

$$a + bi + cj + dk \overset{\psi}{\mapsto} \frac{bi + cj + dk}{1 + a}.$$

where we identify $\mathbb{R}^3 \cong \{0\} \times \mathbb{R}^3 \subset \mathbb{R}^4 \cong Q$ with the strictly imaginary part of the quaternions. So

$$\psi(1) = 0, \qquad \psi(\pm i) = \pm i,, \qquad \psi(\pm j) = \pm j,, \qquad \psi(\pm k) = \pm k.$$

There are $\binom{4}{2} = 6$ coordinate great circles as we take the unit circle in the plane where two coordinates are 0. For $u \neq v \in \{1, i, j, k\}$ two basis vectors the corresponding great circle $\gamma_{u,v} : \mathbb{R} \to S^3$ is

$$t \mapsto u \cos t + v \sin t.$$

We get for $v \neq 1$ that $\psi(\gamma_{1,v}(t)) = v \frac{\sin t}{1+\cos t}$ and for $u, v \neq 1$ we get $\psi(\gamma_{u,v}(t)) = \gamma_{u,v}(t)$. Hence stereographic projection sends the circle $\gamma_{1,v}$ (minus the point -1) for $v \neq 1$ to the line spanned by the basis vector v and for $u, v \neq 1$ the map ψ sends the circle $\gamma_{u,v}$ to itself.

(b) (i) We need to show that $\bigcup_{0\le r\le\pi/2} T_r = S^3$ as a disjoint union. First note that $T_r \subset S^3$ as $|z|^2 = |w|^2 = \cos^2 r + \sin^2 r = 1$. Let now $(z,w) \in S^3$ arbitrary. As $0 \le |z| \le 1$ there is a unique $r \in [0, \pi/2]$ such that $\cos r = |z|$. Moreover, automatically

$$|w| = \sqrt{1-|z|^2} = \sqrt{1-\cos^2 r} = \sin r$$

since $r \in [0, \pi/2]$. Hence $(z,w) \in T_r$. Finally, the T_r are pairwise disjoint for pairwise different r.

(ii) The set $T_0 = \{(z,w) \mid |z| = 1, |w| = 0\}$ is the great circle parametrized by $\gamma_{1,i}$ discussed in part (a). The set $T_{\pi/2} = \{(z,w) \mid |z| = 0, |w| = 1\}$ is the great circle parametrized by $\gamma_{j,k}$. Note that

$$\gamma_{1,i}(t) = \cos t + i \sin t = 1 \cdot e^{it}, \qquad \gamma_{j,k}(t) = j\cos t + k \sin t = j \cdot e^{it}$$

are both Hopf fibers in the sense of exercise 3.

(iii) For $0 < r < \pi/2$ we have a homeomorphism $T_r \to S^1 \times S^1$ sending

$$(z,w) \mapsto \left(\frac{z}{|z|}, \frac{w}{|w|}\right)$$

where $S^1 \subset \mathbb{C}$ is the standard unit circle. Hence T_r is a torus.
An arbitrary point $p \in T_r$ has form

$$\cos r e^{it} + \sin r e^{is} j$$

for some $(s,t) \in \mathbb{R}$. The Hopf fiber through this point is

$$a \mapsto e^{ia} p = \cos r e^{it+a} + \sin r e^{is+a} j$$

which lies again entirely in T_r. They are called *Villarceau circles*. Hence each T_r is the union of Hopf fibers.
For fixed (s,t) the path $\eta_{s,t} : [0, \pi/2] \to S^3$ given by $r \mapsto \cos r e^{it} + \sin r e^{is} j$ is of unit speed and the derivative $\eta'_{s,t}(r)(\eta_{s,t}(r))$ is orthogonal to the torus T_r at $\eta(s,t)(r)$. Using such a path we can get from T_{r_0} to T_{r_1} in time $|r_1 - r_0|$ (independent of which curve $\eta_{s,t}$ we use). As the paths were passing at the same time orthogonally through all other T_r the distance in S^3 from T_{r_0} to T_{r_1} is $|r_1 - r_0|$.

(c) We can parametrize T_r as

$$(s,t) \mapsto \cos r e^{it} + \sin r e^{is} j = \cos r \cos t + i \cos r \sin t + j \sin r \cos s + k \sin r \sin s$$

which under stereographic projection is

$$(s,t) \mapsto \frac{i \cos r \sin t + j \sin r \cos s + k \sin r \sin s}{1 + \cos r \cos t}.$$

We claim that the T_r parametrize a family of standard tori in $\mathbb{R}^3$ which are rotation symmetric with respect to the i-axis. A torus has standard parametrization:

$$(\theta, s) \mapsto i\tilde{r}\sin\theta + j(R + \tilde{r}\cos\theta)\cos s + k(R + \tilde{r}\cos\theta)\sin s$$

where $\tilde{r}$ denotes the radius of the circle in the ij-plane with center at $iR \in \mathbb{R}^3$. Then the torus is the surface of revolution of this circle when rotation around the i-axis. We claim that the following works:

$$\begin{aligned}
\tilde{r} &= \frac{\cos r}{\sin r}, \\
R &= \frac{1}{\sin r}, \\
\sin\theta &= \frac{\sin t \sin r}{1 + \cos r \cos t}, \\
\cos\theta &= \frac{-\cos r - \cos t}{1 + \cos r \cos t}.
\end{aligned}$$

Note that $\sin^2\theta + \cos^2\theta = 1$ for the given formulas.

(d) A nice illustration of preimages of the Hopf fibration

$$S^3 \setminus \{-1\} \to \mathbb{R}^3 \to S^2$$

is given in the following video
`https://www.youtube.com/watch?v=AKotMPGFJYk`

We can define a map

$$S^3 \to S^2$$

that sends $T_r \subset S^3$ to the latitude circles in S^2 at with angular height coordinate r, i.e. the circle T_0 gets mapped to the north pole in S^2 and $T_{\pi/2}$ gets mapped to the south pole in S^2. This defines a smooth fiber bundle with fiber S^1.