Exercise Sheet 8

To be handed in until November 15

1. Unit quaternions and rotations

Let $Q \cong \mathbb{R}^4$ be the quaternions and define the *purely imaginary quaternions* by

$$\mathbb{R}^3 := \{ai + bj + ck \in Q \,|\, a, b, c \in \mathbb{R}\} \cong \{0\} \times \mathbb{R}^3 \subset Q.$$

(a) Verify that the rule

$$Ad_v \colon w \mapsto vwv^{-1}$$

defines an action of the unit quaternions $v \in S^3$ on $\mathbb{R}^3$ by linear isometries (with respect to the usual inner product).

Hint: Use that for $u \in Q$ that u is purely imaginary iff $\overline{u} = -u$.

(b) For any quaternion $u \in Q$ define e^u by power series. Verify that for $n \in S^2 \subset \mathbb{R}^3$ and $\theta \in \mathbb{R}$ we have

$$e^{\theta n} = \cos\theta + n \sin\theta.$$

Show that $e^{\theta n} \in S^3$. Moreover, show that any v in $S^3 \subset Q$ can be written as $v = e^{\theta n}$ for some $\theta \in \mathbb{R}$ and $n \in S^2 \subset \mathbb{R}^3$ (i.e. the exponential $\mathbb{R}^3 \to S^3$ is surjective).

(c) Describe the action of an element v of S^3 on $\mathbb{R}^3$ *geometrically.*

Hint: Ad_v is a rotation by some angle ϕ about some axis. Find the axis and the angle.

(d) Verify that the association $v \mapsto Ad_v$ gives a surjective homomorphism and a two-sheeted covering map from S^3 to $SO(3)$. Consequently, observe that $SO(3) \cong \mathbb{R}P^3$.

Solution:

(a) Let $w \in \mathbb{R}^3$ and $v \in S^3$. To show that Ad_v is well-defined, we need to show that $vwv^{-1} = vw\overline{v}$ is again in $\mathbb{R}^3 \cong \{0\} \times \mathbb{R}^3$. As for the complex numbers: $u \in \mathbb{R}^4$ is in $\mathbb{R}^3$ iff $\overline{u} = -u$. Note that

$$\overline{vw\overline{v}} = \overline{\overline{v}}\,\overline{w}\,\overline{v} = v\,\overline{w}\,\overline{v} = v\,(-w)\,\overline{v} = -vw\overline{v}.$$

Hence $w \in \mathbb{R}^3$ implies $Ad_v(w) = vwv^{-1} \in \mathbb{R}^3$. Moreover, Ad_v is an isometry for every $v \in S^3$ since

$$|Ad_v(w)| = |vwv^{-1}| = |v||w||v|^{-1}.$$

Conjugation defines a group action as $Ad_1 = id_{\mathbb{R}^3}$ and $Ad_u \circ Ad_v = Ad_{uv}$ because

$$Ad_u \circ Ad_v(w) = Ad_u(vwv^{-1}) = uvwv^{-1}u^{-1} = uvw(uv)^{-1} = Ad_{uv}(w).$$

(b) Let $u \in \mathbb{R}^3 \subset Q$ be purely imaginery. To compute the exponential e^u note that using again as in (a) that $\overline{u} = -u$ we have

$$u^2 = -u(-u) = -u\overline{u} = -|u|^2.$$

Hence $u^{2n} = (-1)^n|u|^{2n}$ and $u^{2n+1} = (-1)^n|u|^{2n}u$. So

$$\begin{aligned} e^u &= \sum_{n=0}^{\infty} \frac{u^n}{n!} = \sum_{n=0}^{\infty} \frac{u^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{u^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n|u|^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(-1)^n|u|^{2n}u}{(2n+1)!} \\ &= \cos(|u|) + \sin(|u|)\frac{u}{|u|}. \end{aligned}$$

In particular,

$$|e^u|^2 = \cos^2(|u|) + \sin^2(|u|)\frac{|u|^2}{|u|^2} = 1.$$

So the exponential map $\exp : \mathbb{R}^3 \to S^3$ is well-defined. Moreover, for $u = \theta n$ with $n \in S^2 \subset \mathbb{R}^3, \theta \in \mathbb{R}$ we get

$$v = e^{\theta n} = \cos(\theta) + \sin(\theta)n.$$

The exponential $\mathbb{R}^3 \to S^3$ is surjective: Given a $v \in S^3$ let us find $n \in S^2 \subset \mathbb{R}^3, \theta \in \mathbb{R}$ such that $v = e^{\theta n}$. Writing $v = a + w \in S^3$ with $a \in \mathbb{R}$ real and $w \in \mathbb{R}^3$ purely imaginary we must have $a = \cos(\theta)$ and $w = \sin(\theta)n$. Given (a, w) we can find a unique $\theta \in [0, 2\pi)$ such that $a = \cos\theta$ and then set $n = \frac{w}{\sin}\theta$ for $\theta \neq 0, \pi$ (and $n \in S^2$ aribtrary if $\theta = 0, \pi$).

(c) Let us prove that for $v = e^{\theta n} \in S^3$ that Ad_v is a rotation around the axis $n \in S^2 \subset \mathbb{R}^3$ with angle 2θ.

For $w, \tilde{w} \in \mathbb{R}^3$ and $x, y \in \mathbb{R}$ we have

$$Ad_v(xw + y\tilde{w}) = v(xw + y\tilde{w})\overline{v} = xvw\overline{v} + yv\tilde{w}\overline{v}$$

which proves that $Ad_v : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map.

For $v = e^{\theta n}$ and $w \in \mathbb{R}^3$ we have

$$\begin{aligned} Ad_{e^{n\theta}}(w) &= e^{n\theta}we^{-n\theta} = (\cos\theta + n\sin\theta)w(\cos\theta - n\sin\theta) \\ &= w\cos^2\theta - wn\cos\theta\sin\theta + nw\cos\theta\sin\theta - nwn\sin^2\theta. \end{aligned}$$

To understand what the linear map Ad_v does, it is enough to check what the image of n is and what the image of an element $w \in n^\perp$ is because $\mathbb{R}n \oplus n^\perp = \mathbb{R}^3$.

Claim. For $w = bi + cj + dk, \tilde{w} = fi + gj + hk \in \mathbb{R}^3$ purely imaginery we have

$$w\tilde{w} = -\langle w, \tilde{w}\rangle_{\mathbb{R}^3} + w \times \tilde{w} \in Q.$$

Proof.

$$\begin{aligned} w\tilde{w} &= (bi + cj + dk)(fi + gj + hk) \\ &= -bf - cg - dh + (ch - gd)i + (df - bh)j + (bg - cf)k \\ &= -\langle w, \tilde{w}\rangle_{\mathbb{R}^3} + w \times \tilde{w} \end{aligned}$$

□

Using the formula for Ad_v applied to $w = n$ we get

$$Ad_{e^{n\theta}}(n) = n\cos^2\theta - n^2\cos\theta\sin\theta + n^2\cos\theta\sin\theta - n^3\sin^2\theta = n$$

as by the claim $n^3 = -|n|^2 n = -n$. For $w \in n^\perp$ of unit length let $\tilde{w} = n \times w$ which is automatically also of unit length and $n, w, \tilde{w}$ are an ONB of $\mathbb{R}^3$. Applying the formula for Ad_v to this w yields

$$\begin{aligned} Ad_{e^{n\theta}}(w) &= w\cos^2\theta - wn\cos\theta\sin\theta + nw\cos\theta\sin\theta - nwn\sin^2\theta \\ &= w\cos^2\theta - (w \times n)\cos\theta\sin\theta + (n \times w)\cos\theta\sin\theta - (n \times w)n\sin^2\theta \\ &= w\cos^2\theta + 2\tilde{w}\cos\theta\sin\theta - \tilde{w}n\sin^2\theta \\ &= w\cos^2\theta + 2\tilde{w}\cos\theta\sin\theta - \tilde{w} \times n\sin^2\theta \\ &= w\cos^2\theta + 2\tilde{w}\cos\theta\sin\theta - w\sin^2\theta \\ &= w\cos(2\theta) + \tilde{w}\sin(2\theta). \end{aligned}$$

Hence Ad_v restricted to the subspace $n^\perp$ is a rotation in the plane $n^\perp$ by an angle 2θ. So we proved that $Ad_v : \mathbb{R}^3 \to \mathbb{R}^3$ is a rotation by an angle 2θ with axis n.

(d) As $e^{\theta n} \in S^3$ for any $\theta \in \mathbb{R}$ and $n \in S^2$ we can get any rotation $\mathbb{R}^3 \to \mathbb{R}^3$, hence $S^3 \to SO(3)$ sending $v \mapsto Ad_v$ is surjective. It is also a group homomorphism by part (a). To prove that the map is a two-sheeted covering we can show that its kernel is two elements. Indeed, if $Ad_v(w) = vwv^{-1} = w$ then $v = \pm 1$ as only purely real quaternions commute with any other quaternion. Hence the map $S^3/\{\pm 1\} \cong SO(3)$ is an isomorphism of smooth groups. Because also $S^3/\{\pm 1\} \cong \mathbb{RP}^2$ we also have $SO(3) \cong \mathbb{RP}^2$.

2. Orientation and quotients

(a) Let M be a connected, oriented manifold, and suppose G is a group that acts freely and properly discontinuously on M by diffeomorphisms. Prove that M/G is orientable iff all $g \in G$ are orientation preserving.

(b) Show that $\mathbb{RP}^n$ is orientable iff n is odd.

Solution:

(a) Let $\pi : M \to M/G$ be the projection. Let As π is a local diffeomorphism, $d\pi : T_pM \to T_{\pi(p)}M/G$ is a diffeomorphism, so it takes a basis of T_pM to one of $T_{\pi(p)}M/G$.

If all g are orientation-preserving let us define an orientation at $q \in M/G$ by the image of the orientation for T_pM under π for some $p \in M$ such that $\pi(p) = q$. As $\pi^{-1}(q) = G \cdot p$ and all $g \in G$ are orientation-preserving this defines a well-defined orientation.

Suppose not all g are orientation-preserving and that M is connected. Let us prove by contradiction that M/G is nonorientable. If there is an orientation on M/G then as M is connected either $\pi : M \to M/G$ is orientation-preserving or orientation-reversing. In particular, $d\pi_p : T_pM \to dT_{\pi(p)}M/G$ and $d\pi_{gp} : T_{gp}M \to dT_{\pi(gp)}M/G = dT_{\pi(p)}M/G$ are all either orientation-preserving or orientation-reserving for all $g \in G$. But if there is a g which is orientation-reversing then by the chain rule

$$d\pi_{gp} = d\pi_p \circ dg_{gp}^{-1}.$$

But then $d\pi_{gp}$ and $d\pi_p$ would not be either both orientation-reversing or both orientation-reversing.

(b) Recall that $\mathbb{RP}^n = S^n/\{id, A\}$ where A is the antipodal map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ that sends $(x^0, \ldots, x^n) \mapsto (-x^0, \ldots, -x^n)$. The antipodal map is orientation-preserving iff n is odd. Indeed, A can be written as the composition of $n+1$ reflections. Each reflection reverses the orientation on S^n. As composing an odd number of orientation-reversing maps is orientation-reversing and composing an even number of orientation-reversing maps is orientation-preserving we get the result by applying part (a).

3. Verseuchungsprinzip

Let M be a manifold and $U \subset M$ be open. Prove: If U is nonorientable then M is nonorientable.

Solution:

As $T_pU = T_pM$ for any $p \in U$ any consistent choice of orientations on M produces a consistent choice of orientations on U. More concretely, let $M \to OM$ be a continuous section that chooses an orientation for T_pM for every point

$p \in M$. Then the composition $U \xrightarrow{\text{incl}} M \to OM$ has image in the orientation double cover $OU \subset OM$. This defines a continuous section $U \to OU$.

4. Vector fields on the Klein bottle

Recall that the Klein bottle is $K = \mathbb{R}^2/G$ where G is the group generated by the maps

$$(x, y) \mapsto (x + 1, -y)$$
$$(x, y) \mapsto (x, y + 1).$$

How many pointwise linearly independent vector fields can **you** find on $\mathbb{R}^2/G$?

Solution:

First note that there can be maximal two linearly independent vector fields as the dimension of the Klein bottle is 2. So T_pK is of dimension 2 for any p. The vector field $X(p) = (0, 1)$ on $\mathbb{R}^2$ also defines a vector field on K.

However, the vector field $Y(p) = (1, 0)$ on $\mathbb{R}^2$ does not descend to $K = \mathbb{R}^2/G$. In fact, there are no two linearly independent two vector fields on K. Because in case there are n linearly independent vector fields on a manifold M, then $TM \cong M \times \mathbb{R}^n$ and such M would be orientable. But the Klein bottle K is not orientable, so there are no two linearly independent vector field on K.

5. Orientation with curves

Let M be a smooth manifold.

- **(a)** Let $p, q \in M$ and let $\gamma : [0, 1] \to M$ be a curve connecting p to q. Observe that any chosen orientation O of $T_{\gamma(0)}M$ propagates uniquely along γ to a unique path $O_\gamma(t)$ of orientations of $T_{\gamma(t)}M$ that is "continuous" in t (define this) and $O_\gamma(0)$.
- **(b)** Let γ be a closed curve in M, i.e. $\gamma(0) = \gamma(1)$. We say that γ is *orientation-preserving* if $O_\gamma(0)$ equals $O_\gamma(1)$ (for any choice of $O_\gamma(0)$); otherwise we say that γ is *orientation-reversing*. Show that M is orientable if and only if every closed curve is orientation-preserving.
- **(c)** Conclude that the Möbius strip and the Klein bottle are not orientable.
- **(d*)** (For the ones that know cohomology): Define an element $w_1 \in H^1(M, \mathbb{Z}_2)$ that is measuring the obstruction of M being orientable, i.e. $w_1 = 0$ iff M is orientable. This w_1 is called *the first Stiefel-Whitney class.*

Solution:

(a) Given a chart U such that $TU \overset{\Psi}{\cong} U \times \mathbb{R}^n$ specifying one orientation for T_pM at a point p specifies an orientation for all other points $q \in U$ depending on if the orientation at p is mapped to the standard orientation or the opposite orientation of $\mathbb{R}^n$ via Ψ.

Let $I = [0,1]$ and $\gamma : I \to M$ a smooth curve with $\gamma(0) = p$ and $\gamma(1) = q$. Cover $\gamma(I)$ by charts U_j that have the property $TU_j \cong U_j \times \mathbb{R}^n$. As I is compact also $\gamma(I)$ is compact, so $\gamma(I)$ can be covered by finitely many U_j as specified above. We can order the $U_1, \ldots, U_k$ such that $p \in U_1$ and $q \in U_k$ and $U_j \cap U_{j+1} \neq \emptyset$. Then any orientation for a point in $\gamma(I) \cap U_j$ induces an orientation for all points in $\gamma(I) \cap U_{j+1}$ as the two sets overlap. Given an orientation O at $p = \gamma(0)$ thus defines an orientation for all $\tilde{p} \in \gamma(I)$.

So any smooth curve $\gamma : [0,1] \to M$ and an orientation for $T_{\gamma(0)}M$ defines a unique smooth curve $O_\gamma : [0,1] \to OM$.

(b) A closed loop $I \to M$ is orientation-preserving iff any lift $O_\gamma : I \to OM$ is also a closed loop.

Note that a double cover $\pi : N \to M$ is trivial (i.e. $N \cong M \times \{\pm 1\}$) iff all closed loops have only closed lifts. But we have seen in the lecture that the orientation double cover OM is trivial iff M is orientable.

(c) For the central loop $\gamma : I \to M$ in the Möbius strip M there is no closed lift $\gamma : I \to OM$.

(d) For any loop γ in M define

$$w_1(\gamma) = \begin{cases} 1, & \gamma \text{ reverses orientation,} \\ 0, & \gamma \text{ preserves orientation.} \end{cases}$$

This defines a map $\pi_1(M) \to \mathbb{Z}_2$. As $\mathbb{Z}_2$ is abelian and $H_1(M)$ is the abelianization of $\pi_1(M)$ we get a map $H_1(M) \to \mathbb{Z}^2$ which we can identify with a cohomology class $w_1 \in H^1(M, \mathbb{Z}_2)$.