# Exercise Sheet 9

To be handed in until November 22

## 1. Manifolds are locally compact in the strong sense (Lemma 0.5)

**(a)** Let $M$ be a manifold. Show that for any open set $V \subset M$ and any point $p \in V$ there is an open set $U$ such that

(i) $p \in U$,

(ii) $\overline{U}$ is compact,

(iii) $\overline{U} \subset V$.

Note that this entails that $M$ is locally compact but says more.

**(b)** Find a (non-Hausdorff) space $X$ and $K \subset X$ such that $K$ is compact but $\overline{K}$ is not compact.

**Solution:**

**(a)** Let $V \subset M$ be open and $p \in V$. Let $\psi : U' \to \mathbb{R}^n$ be a chart at $p$ such that $U' \subset V$. As $\psi(U') \subset \mathbb{R}^n$ is open and $\psi(p) \in \psi(U')$ there is a open ball $B_r(\psi(p))$ around $\psi(p)$ which is contained in $\psi(U')$. So also

$$W := B_{r/2}(\psi(p)) \subset \overline{W} = \overline{B_{r/2}(\psi(p))} \subset B_r(\psi(p)) \subset \psi(U')$$

and $\overline{W}$ is compact. Then

$$p \in U := \psi^{-1}(W) \subset \overline{U} = \overline{\psi^{-1}(W)} = \psi^{-1}(\overline{W}) \subset U' \subset V$$

Since $\psi$ is a homeomorphism $U$ is open and $\overline{U}$ is compact as $\overline{W}$ is compact.

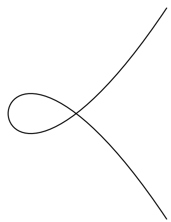
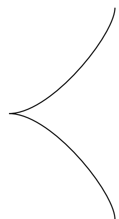
**(b)** Let $X = \mathbb{Z}$ with topology given by $U \subset X$ open iff $0 \in U$ or $U = \emptyset$. Check that this defines a topology. Moreover, finite sets are in any topology. But the closure of $\{0\}$ in this topology is all of $\mathbb{Z}$. Note that

$$\mathbb{Z} = \bigcup_{n \in \mathbb{N}} \{-n, \ldots, n\}$$

is an open cover of $\mathbb{Z}$ that admits no finite subcover. So $\overline{\{0\}} = \mathbb{Z}$ is not compact but $\{0\}$ is in this topology.

**2. Some non-submanifolds**

Prove that the following curves are not submanifolds of $\mathbb{R}^2$:

**(a)**  **(b)** 

**Solution:**

**(a)**

**Claim.** If $X$ and $Y$ are homeomorphic then for any $p \in X$ there is a $q \in Y$ such that $X \setminus \{p\}$ and $Y \setminus \{q\}$ are homeomorphic. In particular, $X \setminus \{p\}$ and $X \setminus \{q\}$ have the same number of connected components.

*Proof.* Any homeomorphism $\Phi : X \to Y$ induces homeomorphisms $\Phi_p : X \setminus \{p\} \to Y \setminus \{\Phi(p)\}$ simply by restriction. □

Suppose the given curve $N$ is a submanifold of $\mathbb{R}^2$. Let $\psi : U \to \mathbb{R}^2$ be a submanifold chart at the problematic point $p$ on the curve $N$ where the crossing happens. By making $U$ smaller we can assume that $U$ is a disk in $\mathbb{R}^2$ and that $U \cap N$ looks like four lines meeting in a point. As $U \cap N$ is homeomorphic to $\psi(U \cap N)$. But for a submanifold chart the image $\psi(U \cap N)$ needs to be an interval in $\mathbb{R} \times \{0\} \subset \mathbb{R}^3$. But an interval and four lines meeting in a point cannot be homeomorphic as deleting one point from an open interval creates always 2 connected components but $\psi(U \cap N \setminus \{p\})$ must have 4 connected components.

**(b)**

**Claim.** Any $m$-submanifold $N \subset \mathbb{R}^n$ admits locally a regular parametrization, i.e. for any $p \in N$ there is an open set $U \subset \mathbb{R}^n$ with $p \in U$, an open set $V \subset \mathbb{R}^m$ a map $\phi : V \to U$ such that $\phi(V) = U \cap N$ and the derivatives $\frac{\partial \phi}{\partial x^1}(q), \ldots, \frac{\partial \phi}{\partial x^m}(q) \in \mathbb{R}^n$ are all linearly independent for all $q \in V$.

*Proof.* Let $\psi : U \to \mathbb{R}^n$ be a sumbanifold chart for $N$ at $p$, i.e. $\psi(U \cap N) \subset \mathbb{R}^m \times \{0\}^{n-m} \subset \mathbb{R}^n$. Set $V := \pi(\psi(U \cap N)) \subset \mathbb{R}^m$ where $\pi : \mathbb{R}^n \to \mathbb{R}^m$ is the projection to the first $m$ coordinates. Note that $V$ is open in $\mathbb{R}^m$. Also, denote by $i : \mathbb{R}^m \to \mathbb{R}^n$ the projection into the first $m$ coordinates. Then $\phi := \psi^{-1} \circ i : V \to \mathbb{R}^n$ is smooth and $\frac{\partial \phi}{\partial x^j}(q) = \frac{\partial}{\partial x^j}$ are the standard vectors for the chart $\phi$ which we know are linearly independent, e.g. by sheet 6 exercise 4. $\square$

We can parametrize regular curves by arc-length. But then if the boundary of the square would admit a parametrization by arc-length then derivative the derivative in for example $x$-direction would not be continuous.

## 3. Nonregular covering spaces

A covering space $\pi : M \to N$ is called *regular* if it comes from a group action, i.e. $N \cong M/G$ for some group $G$ that acts freely and properly discontinuously on $G$. Find a nonregular covering space.

Hint: The base space $N$ needs to have noncommutative fundamental group, e.g. consider the figure 8, or if you want a manifold consider either $\mathbb{C} \setminus \{\pm 1\}$ or the Klein bottle.

**Solution:**

Covering space theory (e.g. `https://pi.math.cornell.edu/~hatcher/AT/AT.pdf` Theorem 1.38, Proposition 1.39) tells us that for a connected manifold, there is a bijection

$$\{\text{covering spaces}\} \longleftrightarrow \{\text{subgroups of } \pi_1(X)\}$$

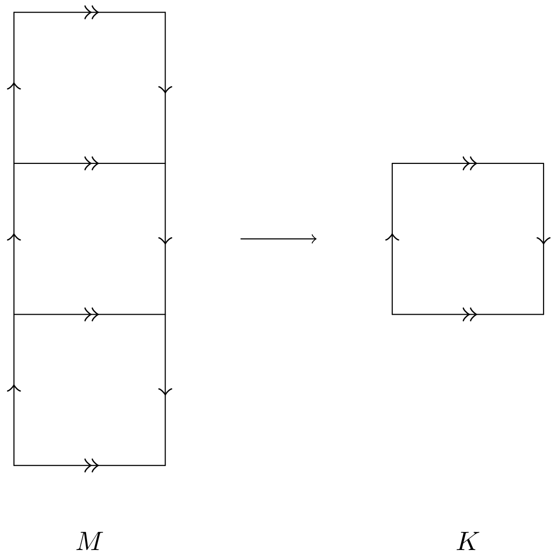
and a bijection

$$\{\text{regular covering spaces}\} \longleftrightarrow \{\text{normal subgroups of } \pi_1(X)\}.$$

The Klein bottle $K$ has fundamental group $\pi_1(K) = \langle a, b \rangle /_{abab^{-1}=1}$. The subgroup $H = \langle a^3, b \rangle \subset \pi_1(K)$ is not normal as conjugating $b \in H$ with $a \in \pi_1(K)$ yields

$$aba^{-1} = a(aba)a^{-1} = a^2 b$$

which is not in $H$. So there must be a nonregular cover $M \to K$ corresponding to the nonnormal subgroup $H$ of $\pi_1(K)$. The following is the (3-sheeted) nonregular covering of the Klein bottle $K$ associated with the subgroup $H$ in $\pi_1(K)$.




Note that $M$ is also a Klein bottle. So we found a 3-sheeted nonregular self-covering map of the Klein bottle.

**4. Another way to get the Hopf fibration**

**(a)** Let $US^2$ denote the *unit tangent bundle* consisting of vectors in $TS^2$ with length 1 (in the usual metric). Show that $US^2$ is diffeomorphic to $SO(3)$.

**(b)** Show that the composition

$$S^3 \xrightarrow{Ad} SO(3) \cong US^2 \xrightarrow{\pi} S^2$$

is equivalent to the Hopf fibration from exercise sheet 7.

**Solution:**

**(a)** There is a bijection

$$US^2 \longleftrightarrow \{\text{ONBs of } \mathbb{R}^3 \text{ that induce the standard orientation}\}.$$

The map is the following. Let $(p, v) \in US^2$, i.e. $p \in S^2$ and $v \in T_pS^2$ with $|v| = 1$. As $p$ and $v$ are orthogonal and have both norm 1. The three vectors $p, v, p \times v$ are an ONB of $\mathbb{R}^3$. The last vector for an ONB (which gives the standard orientation) is uniquely determined by the first two, so the map is indeed, a bijection. This map is actually smooth.

There is also a bijection

$$SO(3) \longleftrightarrow \{\text{ONBs of } \mathbb{R}^3 \text{ that induce the standard orientation}\}.$$

as for a matrix to be in $SO(3)$ means exactly that its column vectors are an ONB of $\mathbb{R}^3$ that induces the standard orientation.

So there is also a diffeomorphism $SO(3) \overset{\Phi}{\cong} US^2$.

**(b)** We need to show that the preimages of a point $p$ in $S^2$ are Hopf circles. The only condition for a matrix $A \in \Phi^{-1}(\pi^{-1}(p)) \in SO(3)$ is that its first column is $p$. The other two columns of $A$ are a (correctly oriented) ONB of $p^\perp$. Suppose $u, u' \in Ad^{-1}(\Phi^{-1}(\pi^{-1}(p))) \in S^3$. To show that they are in the same Hopf circle we need to show that $u' = ue^{it}$ for some $t \in \mathbb{R}$. By definition of $Ad_u$ we have that

$$Ad_u(i) = ui\overline{u} = p = u'i\overline{u}' = Ad_{u'}(i)$$

as $p$ the first column vector, so it is the image of the first basis vector which is $i$. Multiplying the above equation from the left by $\overline{u}$ and from the right by $u'$ yields

$$i\overline{u}u' = \overline{u}u'i.$$

But the only quaternions that commute with $i$ are in $\mathbb{R} \times i\mathbb{R}$. As $\overline{u}u'$ is also of length one we have $\overline{u}u' = e^{it}$ for some $t \in \mathbb{R}$, hence $u' = ue^{it}$. This proves that $u, u'$ are in the same Hopf fiber.

### 5. Equivalent definitions for sizes of the topology of manifolds

Prove that for a manifold $M$ the following are equivalent:

(i) $M$ is second countable.

(ii) $M$ admits a countable atlas.

(iii) $M$ is $\sigma$-compact.

**Solution:**

$(i) \Rightarrow (ii)$: Let $(V_k)_{k\in\mathbb{N}}$ be a countable basis for the topology of $M$. Let $(U_j)_j \in I$ be an atlas. Let $K = \{k \in \mathbb{N} \,|\, V_k \subset U_j \text{ for some } j \in I\} \subset \mathbb{N}$. For every $k \in K$ choose one $U_{j_k}$ such that $V_k \subset U_{j_k}$.

Then $(U_{j_k})_{k\in K}$ is a countable atlas. The indes set $K$ is countable as $K \subset \mathbb{N}$. Also, this atlas covers $M$: Let $p \in M$ be a point. Then $p \in U_j$ for some $j \in I$. As $U_j$ is open, there is a $k \in \mathbb{N}$ such that the basic open set $V_k$ satisfies $p \in V_k \subset U_j$. But then by definition of $K$, we have $k \in K$. So $V_k \subset U_{k_j}$. This proves that $(U_{j_k})_{k\in K}$ covers $M$.

$(ii) \Rightarrow (i)$: If $(U_j)_{j\in\mathbb{N}}$ is a countable atlas of $M$ then each $U_j$ admits a countable basis $(V_k^j)_{k\in\mathbb{N}}$ as it is an open subset of $\mathbb{R}^n$. But then $(V_k^j)_{k\in\mathbb{N},j\in\mathbb{N}}$ is a countable basis of $M$.

$(iii) \Rightarrow (ii)$: Suppose $M$ is $\sigma$-compact and $(U_j)_{j\in I}$ an atlas. Then there is a countable subcover by definition of $\sigma$-compactness (If $M$ is the union of $(K_k)_{k\in\mathbb{N}}$ compact, then each $K_k$ can be covered by finitely many $U_j's$, hence $M$ by countably many $U_j's$).

$(ii) \Rightarrow (iii)$: Suppose $(U_j)_{j\in\mathbb{N}}$ is a countable atlas of $M$. Each $U_j$ is $\sigma$-compact: An open set $U$ in $\mathbb{R}^n$ is $\sigma$-compact as

$$U = \bigcup_{r\in\mathbb{Q}_{>0}, q\in U\cap\mathbb{Q}^n, \overline{B_r(q)}\subset U} \overline{B_r(q)}.$$

As all the $U_j$ are homeomorphic to an open set in $\mathbb{R}^n$, all the $U_j$ are $\sigma$-compact. But the countable union of $\sigma$-compact sets is $\sigma$-compact, so $M$ is $\sigma$-compact.