Supplementary exercises

1. Another topology exercise

(a) Prove that if X, Y are both compact then so is $X \times Y$.

(b) A closed interval $[a, b]$ is compact.

(c) A subset of $\mathbb{R}^n$ is compact if and only it is closed and bounded.

2. Curvatures of parametrized surfaces

Compute H and K for a parametrized surface.

3. Curvatures of graphs

Compute H and K for a graph.

4. Another equivalence of curvatures

Prove directly

$$A(X, Y) = -\langle D_X N, Y \rangle$$

where A is defined by the Hessian of a function f, without going through the expression $\langle D_X Y, N \rangle$.

Hint: Use

$$N = \frac{X_1 \times X_2}{|X_1 \times X_2|},$$

where X_1, X_2 are a basis of $T_p M$ at each point $p \in M$.

5. Curvature and eyeglasses

(a) Interpret your eyeglass prescription as the 2nd fundamental form of a surface.

(b) Research why the true optical situation is more complicated than that.

6. The pseudosphere and the hyperbolic space

(a) Prove that the pseudosphere is locally isometric to hyperbolic space.

(b) Determine via internet the largest known portion of the hyperbolic space that can isometrically embed in $\mathbb{R}^3$.

7. Computing curvature commutes with applying isometries

Let $L : \mathbb{R}^n \to \mathbb{R}^n$ be an isometry.

(a) Let γ be a curve in $\mathbb{R}^n$. Prove

$$k_{L\circ\gamma}(L(\gamma(t))) = L(k_\gamma(\gamma(t))).$$

(b) Let M be a surface in $\mathbb{R}^3$. Prove that

$$A^{L(N)}_{L(M)}(L(p))(L(X), L(Y)) = A^N_M(p)(X, Y)$$

for all $p \in M$ and $X, Y \in T_pM$. The superscripts N and $L(N)$ tell us which normals to use.

8. Locally Euclidean spaces are automatically T_1

Prove that a locally Euclidean space is T_1 (Recall that T_2 = Hausdorff).

9. Existence of admissible pairs

Let M, N be smooth manifolds and $f : M \to N$ a smooth map. Prove: If f is continuous then for all $p \in M$ there is an admissible pair of charts $(U, \psi), (V, \chi)$ with $p \in U$.

10. Is compatibility an equivalence relation?

(a) Is compatibility of charts an equivalence relation? Give an example.

(b) Is compatibility of atlases an equivalence relation?

11. Topological sheaves

A *topological sheaf* is a triple (f, X, Y) such that

i) X, Y are topological space and $f : X \to Y$ is continuous,

ii) f is surjective,

iii) f is a local homeomorphism, i.e. for any $x \in X$ there exists an open neighborhood U of x such that $f(U)$ is open in Y and $f|_U : U \to f(U)$ is a homeomorphism with respect to the induced topologies.

Consider the following example. Let $\mathbb{R}_0 := \mathbb{R}\setminus\{0\}$. Set $Y_1 := \mathbb{R}$, $X_1 := \mathbb{R}_0 \cup \{0^+, 0^-\}$. Let $f_1 : X_1 \to Y_1$ be defined by

$$f_1(x) = \begin{cases} x, & \text{if } x \in \mathbb{R}_0, \\ 0 & \text{if } x \in \{0^+, 0^-\}. \end{cases}$$

Let $\mathcal{T}_{X_1}$ be the topology given by

$$\mathcal{T}_{X_1} = \{U \subset X_1 \mid f_1(U) \text{ is open in } Y_1\}$$

Observe that

(a) f_1 is a topological sheaf and the maps $\phi_\pm := f_1|_{\mathbb{R}_0 \cup \{0^\pm\}}$ define a smooth atlas on X_1, but the induced topology is not Hausdorff.

(b) More generally, any topological sheaf $f : X \to \mathbb{R}^n$ automatically acquires a smooth atlas consisting of its local homeomorphisms onto open subsets of $\mathbb{R}^n$.

(c*) The sheaf of germs of holomorphic functions over $\mathbb{C}$ is Hausdorff and is a smooth manifold. The sheaf of germs of smooth real-valued functions over $\mathbb{R}$ is an extreme example of *non-Hausdorff manifold*.

12. Curves that agree up to order n on manifolds

(a) Let γ be a curve on a manifold M with $\gamma(0) = p$. Write $\gamma(t) = \gamma(0) + O_\psi(|t|^k)$ if for a chart (U, ψ) with $p \in U$ we have

$$\widetilde{\gamma}(t) = \widetilde{\gamma}(0) + O(|t|^k)$$

where $\widetilde{\gamma} = \psi \circ \gamma$.

Prove: We get the same k for any chart, i.e. for all charts ψ_1, ψ_2 we have

$$\gamma(t) = \gamma(0) + O_{\psi_1}(|t|^k) \qquad \text{iff} \qquad \gamma(t) = \gamma(0) + O_{\psi_2}(|t|^k)$$

(b) Let α, β be smooth curves in $\mathbb{R}^n$. We say that α, β *agree up to order n at $t = 0$* if the first $n+1$ terms of the Taylor expansion agree at $t = 0$, i.e.

$$\left(\frac{d}{dt}\right)^k \alpha(0) = \left(\frac{d}{dt}\right)^k \beta(0)$$

for all $k = 0, \ldots n$. Prove that this condition can be meaningfully translated to smooth manifolds.

13. Complex structures and the space of orientable hyperplanes

Let $\tilde{G}(n,k)$ be the space of oriented k-planes through 0 in $\mathbb{R}^n$. Show that $\tilde{G}(4,2) \cong S^2 \times S^2$ as follows:

We call $J : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ a *complex structure* compatible with the Euclidean metric if $J^2 = -id$ and J is an isometry. $(\mathbb{R}^{2N}, J)$ becomes a complex vector space isomorphic to $\mathbb{C}^N$. Note that J induces an orientation on $\mathbb{R}^{2N}$ via the real basis $e_1, Je_1, \ldots, e_N, Je_N$, where $e_1, \ldots, e_N$ is any complex basis of $(\mathbb{R}^{2N}, J)$. Let $\mathcal{J}_0(\mathbb{R}^{2N})$ be the complex structures that induce the standard orientation and $\mathcal{J}_1(\mathbb{R}^{2N})$ be those that induce the opposite orientation.

(a) Show $\mathcal{J}_0(\mathbb{R}^4)$ and $\mathcal{J}_1(\mathbb{R}^4)$ are both diffeomorphic to S^2.

(b) Show $\tilde{G}(4,2)$ is diffeomorphic to $\mathcal{J}_0(\mathbb{R}^4) \times \mathcal{J}_1(\mathbb{R}^4)$.

14. The rotation vector field

Express the vector field on S^2 that rotates along the circles of latitude in polar coordinates on $S^2 \setminus \{N, S\}$.

15. Equivalence of tangent vectors and derivations

Consider the following alternative version definition of a tangent vector: a tangent vector to M at p is a pair (p, Y) where Y is a *derivation at* p, meaning that Y is a linear map

$$Y : C^\infty(M) \to \mathbb{R}, \qquad u \mapsto Y \cdot u,$$

that satisfies the Leibniz rule at p:

$$Y \cdot (uv) = (Y \cdot u)\, v(p) + u(p)\, (Y \cdot v), \qquad u, v \in C^\infty(M).$$

1. Prove that a tangent vector at p (as defined in class) is a derivation at p.
2. Prove that a derivation at p is a tangent vectorat p (as defined in class).

Hint: Let Y be a derivation at p. We will show that Y may be expressed as a linear combination of $\left(\partial/\partial x^1\right)_{p,\psi}, \ldots, \left(\partial/\partial x^n\right)_{p,\psi}$.

i) Let $\psi = (\psi^1, \ldots, \psi^n) : U \to \mathbb{R}^n$ be a chart with $p \in U$. Let χ be a cutoff function for p in U that is constant in a neighborhood of p. For each $i = 1, \ldots, n$, define a special cut-off coordinate function on M by $\phi^i(x) := \chi(x)\psi^i(x)$ for $x \in U$, and extend ϕ^i by zero on the rest of M. Check that $\phi^i \in C^\infty(M)$.

ii) Define X in T_pM by

$$X := \sum_{i=1}^{n} X^i \left(\frac{\partial}{\partial x^i}\right)_{p,\psi}$$

where $X^i := Y \cdot \phi^i$. Prove: $Y \cdot \phi^i = X \cdot \phi^i$ for $i = 1, \ldots, n$.

iii) Prove that $Y \cdot u = X \cdot u$ for any u in $C^\infty(M)$, so Y belongs to T_pM. Hint: use a special version of the Taylor expansion with remainder to show that u may be written as $u(q) = u(p) + \sum_i a_i \phi^i(q) + \sum_i g_i(q)\phi^i(q)$, where a_i are constants and each g_i vanishes at p. Then use the fact that Y is a derivation at p. (See Lee: *Introduction to Smooth Manifolds.*)

16. Local diffeomorphisms

(a) Let $f : \mathbb{R}^1 \to \mathbb{R}^1$ be a local diffeomorphism. Prove that its image is an open interval and that $f\colon \mathbb{R}^1 \to f(\mathbb{R}^1)$ is a diffeomorphism.

(b) Find a function $f : \mathbb{R}^2 \to \mathbb{R}^2$ that is a local diffeomorphism, but is not a diffeomorphism onto its image.

(c) Show that a local diffeomorphism from a compact manifold to another manifold of the same dimension is a covering map.

17. On the degree of covering maps

Let $f\colon M \to N$ be a covering map and N connected.

(a) Show that the number k of elements in $f^{-1}(q)$ is constant on N. (We call f a *k-sheeted* covering).

(b) How many "different" 3-sheeted coverings can you find over S^1?

18. Some immersions and submersions

(a) Prove that the map $\mathbb{R} \to \mathbb{R}^2$ defined by $t \mapsto (t^2, t^3 - t)$ is an immersion.

(b) Prove that the map $\mathbb{R}^{n+1} \setminus \{0\} \to S^n$ defined by $x \mapsto \frac{x}{|x|}$ is a submersion.

19. The Veronese embedding is an embedding

(a) Express the image of the Veronese embedding $f : \mathbb{RP}^2 \to \mathbb{R}^4$ as the zero set of polynomials.

(b) Conclude that $f(\mathbb{RP}^2)$ is a submanifold.

20. Orientation-preserving/reversing maps

(a) Prove that a local diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ is orientation-preserving iff $\det(df(p)) > 0$ for all $p \in M$.

(b) Prove that the composition of two orientation-preserving maps is orientation-preserving.

(c) Prove that the composition of two orientation-reversing maps is orientation-preserving.

(d) Prove that the composition of one orientation-reversing map and an orientation-preserving map is orientation-reversing.

(e) Let $f : M \to N$ be a local diffeomorphism. Prove: if M is connected then f is either orientation-preserving or orientation-reversing.

(f) Give an example of a map that is neither orientation-preserving nor orientation-reversing.

21. Orientation and products

Prove:

(a) The product of two orientable manifolds is orientable.

(b) The product of any manifold with a nonorientable manifold is nonorientable.

22. Orientation and disjoint unions

Let $M = \bigcup_{j=1}^{m} M_j$ be the disjoint union of connected components M_j. How many orientations does M have?

23. Orientable iff orientation cover is trivial

Prove: A manifold M is orientable iff the orientation cover OM is diffeomorphic to the product manifold $M \times \{\pm 1\}$ where $O_p M$ goes to $\{p\} \times \{\pm 1\}$.

24. Orientation and subatlases

(a) Show that an orientation of a smooth manifold $(M, \tau, \overline{\mathcal{A}})$ can be given by specifying a subatlas of $\mathcal{A}$ whose overlap maps are orientation-preserving, and vice versa.

(b) Show that there is a 1-1 correspondence between

(i) orientations on M,

(ii) maximal subatlases $\overline{\mathcal{A}}$ subject to the condition that the overlapping maps are orientation-preserving.

25. The orientation double cover

(a) Prove that OM has the structure of a smooth manifold in a natural way.

(b) Prove that OM is orientable.

26. Lemma 0

Let X be a topological space. Prove that if $C \subset X$ is closed and $K \subset X$ is compact then $C \cap K$ is compact.

27. Lemma 1

A manifold is locally compact.

28. Lemma 2

Let Y be a locally compact Hausdorff space and $D \subset Y$. Then D is closed iff for all $K \subset Y$ compact also $K \cap Y$ is compact.

29. Lemma 3

Let X be a topological space, Y a locally compact Hausdorff space and $f : X \to Y$ proper. Then $f(X) \subset Y$ is closed.

30. Theorem 1

Prove that an embedding is proper iff its image is closed.

31. Proper is necessary in Theorem 1

Give an example of an embedding that is not proper. Observe that the image is not closed.

32. Theorem 2

Prove that a proper injective immersion is closed.

33. The embedding criterion: Theorem H

If an immersion is a homeomorphism onto its image then it is an embedding.

34. Exercise Z

Prove: If M is compact and $f : M \to N$ an injective immersion, then f is an embedding.

35. Hausdorff is necessary in Exercise Z

Find a counterexample to Exercise Z when N is not Hausdorff.

36. * Embeddings of $\mathbb{RP}^3$

Find an embedding of $\mathbb{RP}^3$ into $\mathbb{R}^6$. Does there exist an embedding into $\mathbb{R}^5$ or even into $\mathbb{R}^4$?

37. Second countable manifold are paracompact

Prove that a second countable manifold is paracompact.

38. Connected paracompact manifolds are second countable

Prove that a paracompact connected manifold is second countable.

39. Connected paracompact manifolds are second countable

Prove that a second countable space is separable.

40. Connectedness for manifolds

Prove that a manifold is connected iff it is path-connected.

41. Second countability is preserved for submanifolds

Let M be a second countable manifold. Prove that every submanifold of M is also second countable.

42. Subsets and σ-compactness

(a) Prove that every open or closed subset of $\mathbb{R}^n$ is σ-compact.

(b) Find an subset of $\mathbb{R}$ that is not σ-compact.

43. Transverse intersections are submanifolds

Let P^p, Q^q be submanifolds of a manifold M^m. We call P and Q *transverse* (written $P \pitchfork Q$) if for all $p \in P \cap Q$ we have

$$T_pP + T_pQ = T_pM.$$

Prove: If $P \pitchfork Q$ then $P \cap Q$ is a submanifold of M of dimension

$$2m - p - q$$

or $P \cap Q$ is empty.

44. Intersecting transversely is a generic condition

Let P, Q be submanifolds of a manifold M. Prove that there is a small perturbation $\widetilde{P}$ of P such that $\widetilde{P} \cap Q$ is a submanifold of M.
Hint: Prove it first for P, Q compact submanifolds of $\mathbb{R}^n$.

45. Another version of a bump function for open sets

Let $U \subset \mathbb{R}^n$ be open. Construct a function $\mathbb{R}^n : U \to [0, \infty)$ which is smooth and $f^{-1}(0, \infty) = U$.

46. Main lemma to get existence of partition of unity

Suppose

1. M is a paracompact manifold,
2. $\mathcal{O}$ a cover and
3. $\mathcal{B}$ a subbase.

Then there is a cover $\mathcal{P}$ such that

(i) $\mathcal{P} << \mathcal{O}$,

(ii) $\mathcal{P}$ locally finite,

(iii) $\mathcal{P} \subset \mathcal{B}$.

47. Functions with a lot of critical points and values

(a) Give an example of a smooth function $\mathbb{R} \to \mathbb{R}$ where the critical values are both dense in $[0, 1]$.

(b) Is there a such an example of a function with compact domain?

48. Local vector operations

Let X, Y be smooth vector field on a manifold M and u a smooth function. Prove that the following operations are local:

(i) $(X, u) \mapsto X \cdot u$,

(ii) $(X, Y) \mapsto [X, Y]$.

By this we mean that for all open sets $U \subset M$ and vector fields X_1, X_2, Y_1, Y_2 and functions u_1, u_2 on M:

(i) If $X_1|_U = X_2|_U$ and $u_1|_U = u_2|_U$ then $(X_1 \cdot u_1)|_U = (X_2 \cdot u_2)|_U$,

(ii) If $X_1|_U = X_2|_U$ and $Y_1|_U = Y_2|_U$ then $([X_1, Y_1])|_U = ([X_2, Y_2])|_U$.

49. Vector fields on the long line

Let L be the long line. Prove: there is no vector field X on L such that $X(p) \neq 0$ for all p in L.

50. Compact sets can always be flowed for positive time

Let X be a smooth vector field on a manifold M and $K \subset M$ a compact set. Prove that there is an open set $U \supset K$ and a $\delta > 0$ such that the flow

$$\phi_X : U \times (-\delta, \delta) \to M$$

is well-defined.

51. Extending the flow on open sets

Let M be a manifold and X a smooth vector field on M.

(a) Fix $x \in M$. Suppose the flow $\phi_X^s(x)$ exists for $0 \leq s \leq t$. Prove that there exists an open set U containing x and a $\delta > 0$ such that

$$\phi_X : U \times [0, t + \delta] \to M$$

exists.

(b) Conclude that the set $\mathcal{U}$ (the maximal set in $M \times \mathbb{R}$ where where the flow exists) is open.

(c) Suppose $\phi_X : U \times [0, t] \to M$ exists where $U \subset\subset M$ is open. Is it true that there is an open set V and a $\delta > 0$ such that $U \subset\subset V \subset\subset M$ and

$$\phi_X : V \times [0, t + \delta] \to M$$

exists?

52. Properties of the Lie derivative

Consider the following statements:

(i) $L_X Y$ is linear in X and Y.

(ii) $L_X Y = -L_Y X$ and $L_X X = 0$.

(iii) $L_X(gY) = gL_X Y + (X \cdot g)Y$ and $L_{fX} Y = fL_X - (Y \cdot f)X$.

All are consequences of $L_X Y = [X, Y]$ and the properties of $[X, Y]$.

(a) Which of these statements can be proven directly from the definition of $L_X Y$ without using $L_X Y = [X, Y]$?

(b) Which of these statements can be proven from each other?

53. Pulling back by the flow for all times

Let ϕ_X^t be the flow of a vector field X, and Y another vector field.

(a) Prove
$$\frac{d}{ds}(\phi_X^s)^*(Y) = (\phi_X^s)^*(L_X Y)$$

(b) Recalling $L_X Y = [X, Y]$, use this to give another prove that
$$\phi_X^t(X) = X.$$

54. Explicit computations of Lie derivatives

(a) Compute from the definition
$$L_{x\frac{\partial}{\partial x}}\left(\frac{\partial}{\partial y}\right), \qquad L_{\frac{\partial}{\partial y}}\left(x\frac{\partial}{\partial x}\right),$$
$$L_{y\frac{\partial}{\partial x}}\left(\frac{\partial}{\partial y}\right), \qquad L_{\frac{\partial}{\partial y}}\left(y\frac{\partial}{\partial x}\right).$$

(b) Attempt to make drawings of the above effects. Which flows are shears? Anisotropic expansions? Isometries?

55. Standard vector fields of charts commute

Verify that
$$\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0.$$

56. Partially coordinate vector fields.

(a) Find an example of a 2-manifold M and vector fields X, Y such that X, Y are coordinate vector fields on part but not all of M.

(b) Find such examples with linear vector fields.

57. Commuting flows of left-invariant vector fields

Let $G = GL(n, \mathbb{R})$ and $A \in \mathbb{R}^{n\times n}$. Let $Y_A \in C^\infty(TG)$ be the left-invariant vector field
$$Y_A(C) = CA, \qquad \text{for } C \in G.$$

(a) Compute its flow $\phi^t_{Y_A}$.

(b) Prove that $\phi^t_{Y_A} \circ \phi^t_{Y_B} = \phi^t_{Y_B} \circ \phi^t_{Y_A}$ iff $[Y_A, Y_B] = 0$ directly using matrix methods.

Hint: Use that $[Y_A, Y_B] = Y_{[A,B]}$.