Differential Geometry
Lecture held by Prof. Ilmanen

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Warning: We are sure there are lots of mistakes in these notes. Use at your own risk! Corrections and other feedback would be greatly appreciated and can be sent to `mitschriften@vmp.ethz.ch`. If you report an error please always state what version (the first number on the `Id` line below) you found it in. For further information see:

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Id

Contents

1 Introduction: curves and surfaces

Riemannian Geometry is a subset of Differential Geometry
A *Riemannian manifold* is a *smooth manifold* endowed with a notion of *(infinitesimal) arclength* → *Riemannian metric*: $g = g_{ij}(x)dx^i dx^j$

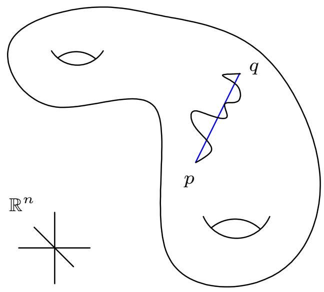


Figure 1: A Riemannian manifold is endowed with a notion of infinitesimal acrlength, thus a shortest path (a *geodesic*) can be defined between two points on the manifold.

Curvature

extrinsic curvature $M^k \subset \mathbb{R}^n$	intrinsic curvature
how M curves inside $\mathbb{R}^n$	how M curves "inside itself"

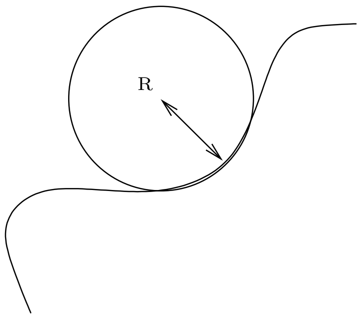


Figure 2: The radius of curvature is the radius of the circle which most closly approximates the curve at a given point.

Doing calculus on the manifold

$$D_i f, \quad D_i D_j X^k \neq D_j D_i X^k, \qquad X \text{ a vector field}$$

Derivatives can't be commuted arbitrarily

$$D_i D_j X^k = D_j D_i X^k + R_{ij\ell}{}^k X^\ell$$

where R is the Riemannian curvature tensor.

1.1 Curves in Space

Basic notation:

$$\mathbb{R}^n,\ x = (x^1, \ldots, x^n)$$

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathbb{R}^n}$$

$$|x|_{\mathbb{R}^n} := \langle x, x \rangle_{\mathbb{R}^n}^{\frac{1}{2}}$$

A *regular curve* is a smooth (= infinitely differentiable = C^∞) function

$$\gamma : [a, b] \to \mathbb{R}^n,$$

such that $\frac{d\gamma}{dt} \neq 0\,\forall t$

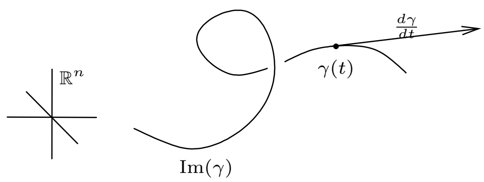


Figure 3: A regular curve and its velocity vector (derivative).

Example of a non regular curve:

$$t \mapsto (t^2, t^3) \in \mathbb{R}^2$$

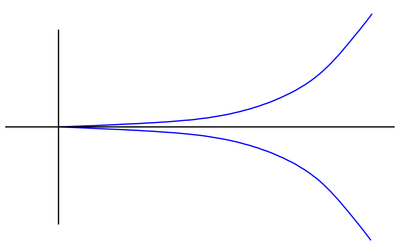


Figure 4: A curve whose derivative vanishes at 0 and is thus not regular.

Arclength

$$s(t) := \int_{t_o}^{t} \left| \frac{d\gamma}{dt} \right| dt$$

Reparameterize by arclength, get

$$\gamma = \gamma(s), \left| \frac{d\gamma}{ds} \right| = 1$$

Unit Tangent Vector

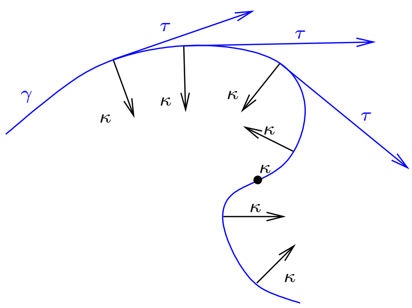


Figure 5: A curve parametrized by arclength always has a tangent vector of unit length.

$$\tau(s) := \frac{d\gamma}{ds} = \frac{d\gamma/dt}{|d\gamma/dt|}$$

Definition *the curvature vector κ of γ at s is*

$$\kappa(s) := \frac{d\tau}{ds} = \frac{d^2\gamma}{ds^2} \in \mathbb{R}^n$$

Proposition 1.1 $\kappa \perp \tau$

Proof

$$\langle \tau, \tau \rangle = 1$$

$$0 = \frac{d}{ds}\langle \tau, \tau \rangle = 2\langle \frac{d\tau}{ds}, \tau \rangle = 2\langle \kappa, \tau \rangle$$

□

Exercise: Show for $\gamma(t)$ (not necessarily parametrized by arclength)

$$\kappa = \frac{1}{|\gamma_t|^2}\left(\gamma_{tt} - \left\langle \gamma_{tt}, \frac{\gamma_t}{|\gamma_t|} \right\rangle \frac{\gamma_t}{|\gamma_t|} \right)$$

Curves in $\mathbb{R}^2$

κ reduces to a number k. Define k by $\kappa = kN$ (*curvature as a scalar*)

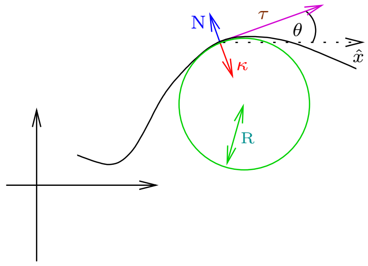


Figure 6: For kurves in the plane curvature reduces to a number k.

We can show:

$$\begin{aligned} k &= \frac{1}{R} && R := \text{radius of curvature, i.e. radius of osculating circle} \\ &= \frac{d\theta}{ds} && \theta := \text{angle between } \tau \text{ and x-axis} \\ &= \frac{u_{xx}}{(1+u_x^2)^{3/2}} && \text{If we write } \gamma \text{ as } y = u(x) \end{aligned}$$

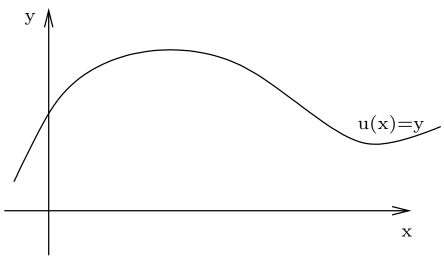


Figure 7: The curve γ defined as a graph $y = u(x)$.

Theorem 1.2 *$k(s)$ determines γ up to a rigid motion of $\mathbb{R}^2$ (to make the starting point $\gamma(0)$ and starting direction $\gamma_s(0)$ coincide, see figure 8).*

Curves in $\mathbb{R}^3$

If $\kappa \neq 0 \, \forall t$ we call γ an *ordinary curve* and define

$$\begin{aligned} N &:= \tfrac{\kappa}{|\kappa|} && \text{normal } (\perp \tau) \\ k &:= |\kappa| && \text{curvature scalar (note } k > 0) \\ B &:= \tau \times N && \text{binormal} \end{aligned}$$

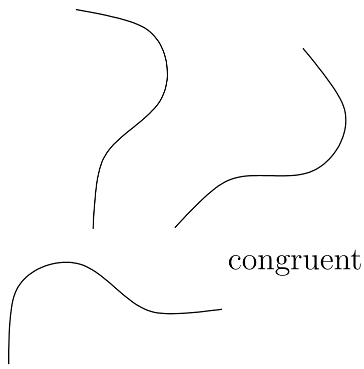


Figure 8: Congruent lines which differ only by rigid motion.

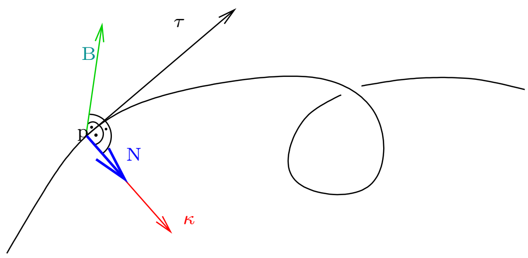


Figure 9: In 3 dimensions κ can move more freely, so a skalar is no longer enough to describe it.

(τ, N, B) orthonormal basis along γ, called a *moving frame*

Definition
Torsion vector:

$$\lambda := \langle \frac{dN}{ds}, B \rangle B \in \mathbb{R}^3$$

torsion scalar:

$$\ell := \langle \frac{dN}{ds}, B \rangle \in \mathbb{R}$$

λ is the measure of that portion of the change of N that occurs within the 2-dimensional normal plane spanned by N, B (That is captured by κ and not that part due to the turning of the normal plane itself.

$k(t)$ is a "2nd derivative of γ" and ℓ is a "3rd derivative"
Exercise

i. Compute k, ℓ at $t = 0$ for $t \to (t, at^2, bt^3)$

ii. If the torsion $\ell \equiv 0$, show γ lies in a plane.

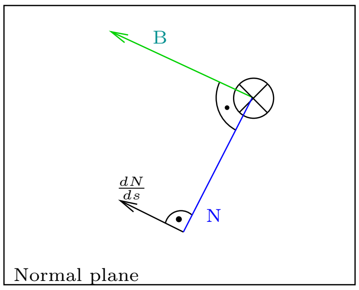


Figure 10: Torsion

iii. If k and ℓ are constant along γ, prove γ is a helix.

iv. * Prove theorem 1.3.

Theorem 1.3 *Any given smooth functions $k(s) > 0$, and $\ell(s)$ of arclength determine γ in $\mathbb{R}^3$ uniquely, up to a rigid motion (isometry) of $\mathbb{R}^3$*

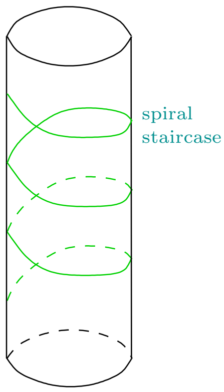


Figure 11: A curve of constant torsion and curvature is a helix (spiral staircase).

Some Global Theorems

local (infinitesimal)	$\longleftrightarrow$	global
curvature measures local geometry		integral quantities
		topology

γ is called *simple* (or *embedded*) if γ has no self intersections

γ is called closed if $\gamma : [a, b] \to \mathbb{R}^n,\ \gamma(a) = \gamma(b)$

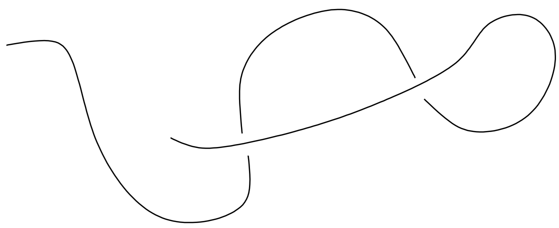


Figure 12: A curve with self intersections, which is therefore not simple.

Theorem 1.4 *γ closed curve in $\mathbb{R}^2$. Then:*

i. $\int_\gamma k ds = 2\pi n \quad \exists n \in \mathbb{Z}$

ii. If γ is simple, then $n = \pm 1$

Proof i.

$$\int_\gamma k \quad ds = \int_a^b k \quad ds = \int_a^b \frac{d\theta}{ds} ds = \theta(b) - \theta(a) \in 2\pi\mathbb{Z}$$

θ is well defined on $\mathbb{R}$, with

$$\theta(s) = \theta(s + b - a) + 2\pi n \qquad \exists n$$

□

Theorem 1.5 *γ closed curve in $\mathbb{R}^3$. Then*

i.

$$\int_\gamma |\kappa| \, ds \geq 2\pi$$

ii. (Milnor) If γ is knotted then

$$\int_\gamma |\kappa| \, ds \geq 4\pi$$

This yields a relation between global integrals and global topology.

1.2 The Geometry of Surfaces in $\mathbb{R}^3$

T_pM is the *tangent space* of vectors tangent to M at p and $N \equiv N(p)$ is a unit normal to M at p

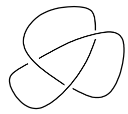


Figure 13: A *knotted* curve wich *cannot* be deformed to the standard circle without developing self intersections.

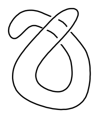


Figure 14: an *unknotted* curve which can be deformed to standard circle without developing self-intersections

1.2.1 (Extrinsic) Curvature

κ is the curvature vector of γ

$$\kappa = kN \ \exists k \in \mathbb{R}$$

Compute k: Choose orthonormal coordinates in $\mathbb{R}^3$ such that

$$p = (0,0,0)$$

$$T_pM = xy\text{-plane (i.e. } M \text{ is tangent to the } xy\text{-plane at } p)$$
$$N = (0,0,1) \text{ (i.e. } N \text{ points in the positive } z\text{-direction)}$$

Note Then M is the graph (locally) of some function $z = f(x,y)$ such that

$$f(0,0) = 0, \ \left.\frac{\partial f}{\partial x}\right|_{0,0} = \left.\frac{\partial f}{\partial y}\right|_{0,0} = 0$$

P is spanned by N, v where v is some unit vector in the xy-plane, $v = (v^1, v^2, 0)$.

Claim The curvature of γ is

$$k = \begin{pmatrix} v^1 & v^2 \end{pmatrix} \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(p) & \frac{\partial^2 f}{\partial x \partial y}(p) \\ \frac{\partial^2 f}{\partial x \partial y}(p) & \frac{\partial^2 f}{\partial y^2}(p) \end{pmatrix} \begin{pmatrix} v^1 \\ v^2 \end{pmatrix} = v^T D^2 f(p) v$$

with $D^2 f(p)$ being the Hessian of f at p

Proof Give P orthogonal coordinates (u, z). In these coordinates, γ is then given by

$$\begin{aligned} z &= g(u) := f(uv^1, uv^2) \\ g(0) &= g_u(0) = 0 \end{aligned}$$

$$k(0) = \left. \frac{g_{uu}}{(1+g_u^2)^{3/2}} \right|_0 = g_{uu}(0)$$

Use chain rule on $g = f \circ (u \mapsto (uv^1, uv^2))$.

$\square$

The bilinear form $(D^2 f)_p$ is called the *second fundamental form* or *extrinsic curvature tensor* of M at p. Written:

$$A(p)(\text{or } II(p)) : T_pM \times T_pM \to \mathbb{R}$$

Warning The Hessian formula for $A(p)$ is valid only when

$$\left. \frac{\partial f}{\partial x} \right|_{0,0} = \left. \frac{\partial f}{\partial y} \right|_{0,0} = 0$$

Exercise

Suppose M is given as a graph $z = f(x, y)$. Find a formula for $A(p)$ with respect to the coordinates on T_pM given by x, y.

Find an analogous formula for the case of a parametrized surface

$$\phi : \mathbb{R}^2 \supset U \to V \subseteq M \subseteq \mathbb{R}^3$$

U, V open, ϕ smooth with injective differential.

We can rotate the xy-plane so that $A(p)$ becomes diagonal:

$$A(p) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

k_1 and k_2 really capture the geometry of the surface

Definition k_1, k_2: *principal curvatures* of M at p

$$\begin{aligned} H &:= k_1 + k_2 : \quad \textit{mean curvature of } M \textit{ at } p \\ K &:= k_1 k_2 = \det A : \quad \textit{Gauss curvature of } M \textit{ at } p \end{aligned}$$

Examples

Sphere of radius R has

$$\begin{aligned} k_1 = \ & k_2 \ = \frac{1}{R} \\ K = \ & \tfrac{1}{R^2} \\ H = \ & \tfrac{2}{R} \end{aligned}$$

Cylinder of radius R has eigenvectors $e_1,\ e_2$, where e_1 points along the cylinders' axis and e_2 is tangent to the circle that goes around the cylinder, and eigenvalues $k_1 = 0,\ k_2 = \frac{1}{R}$

$$H = \frac{1}{R},\ K = 0 \cdot \frac{1}{R} = 0$$

Catenoid C:

It is the rotation of curve $\gamma : y = \cosh x$ around the x-axis. Let e_1 be tangent to γ and e_2 tangent to a circle of rotation.

The eigenspaces of A are preserved by the reflections R_Q across planes $Q \supseteq x$-axis. Thus the eigenvectors of A must be e_1, e_2 (since these are the only directions preserved by R_Q). So evidently $k_1 > 0 > k_2$ if N is outward.

Compute $k_1 = A(e_1, e_1) =$ curvature of γ, the graph of $g(x) = \cosh x$

$$k_1 = \frac{g_{xx}}{(1+g_x^2)^{3/2}} = \frac{\cosh x}{\cosh^3 x} = \frac{1}{\cosh^2 x}$$

Exercise Compute that $k_2 = -\frac{1}{\cosh^2 x}$. Then

$$H = \frac{1}{\cosh^2 x} - \frac{1}{\cosh^2 x} = 0$$

We call a surface of equal and opposite curvatures *minimal surface*

Exercise (Helicoid)

Let L_1 be a vertical line and let L_2 be a line normal to L_1 Move L_2 upward at constant speed while rotating slowly about the point of intersection with L_1.

Prove $H = 0$, compute K

1.2.2 Intrinsic Geometry

Let $M \subseteq \mathbb{R}^3$.

$$\gamma : [a, b] \to M$$

$$\gamma(a) = p,\ \gamma(b) = q$$

Length:

$$L(\gamma) := \int_a^b \langle \dot\gamma(t), \dot\gamma(t)\rangle_{\mathbb{R}^3}^{1/2}\, dt$$

Intrinsic distance in M

$$d_M(p, q) := \inf\{\, L(\gamma) |\, \gamma(a) = p, \gamma(b) = q\}$$

(M, d_M) metric space (please verify)

Geodesic:

a curve that *locally* minimizes length (and therefore: *realizes* distance)

Example Sphere: an arc of a great circle minimizes length if it has length less than πR, but is a geodesic even if it is longer.

Riemannian metric of M

Restrict $\langle\cdot,\cdot\rangle_{\mathbb{R}^3}$ to T_pM:

$$\langle X, Y\rangle_{M,p} := \langle X, Y\rangle_{\mathbb{R}^3} \qquad\qquad Y, X \in T_pM$$

Write $g(p) \equiv \langle\cdot,\cdot\rangle_{M,p} : T_pM \times T_pM \to \mathbb{R}$, a positive definite symmetric bilinear form that determines $L(\cdot)$ and $d_M(\cdot,\cdot)$

Definition A property of M is *intrinsic* if it depends only on g.

Isometries

A bijection $\phi : M \to N$ is called an *isometry* if it preserves the metric, i.e.

$$d_M(p, q) = d_N(\tilde p, \tilde q) \qquad \text{, where } \phi(p) = \tilde p, \phi(q) = \tilde q,$$

or

$$g_M(p)(X, Y) = g_N(\tilde p)(\tilde X, \tilde Y) \qquad \text{, where } \phi \text{ takes } X \text{ to } \tilde X \text{ and } Y \text{ to } \tilde Y.$$

(infinitesimal version)

Definition A property (quantity, tensor, structure, etc) is called *intrinsic* if it is preserved by isometries.

Example The rolling map from the flat plane to the cylinder is a *local* isometry (i.e. each point has a neighborhood U such that $\phi|U : U \to \phi(U)$ is an isometry.

We see from the example that

k_1, k_2 are *not* intrinsic

$H(:= k_1 + k_2)$ is *not* intrinsic

Example Cone: Also locally isometric to the plane.

Definition A *developable surface* is a surface in $\mathbb{R}^3$ that is local isometric to a plane.

Example ping-pong ball (hemisphere): it can be deformed in space in such a way that it remains isometric to the original hemisphere (the material does not stretch!).

Exercise Show that the catenoid and helicoid are locally isometric!

A local theorem

Theorem 1.6 *(*Theorema Egregium*) K (the Gauss curvature) is intrinsic!*

There is an intrinsic characterization of K:

$$A(r) = \pi r^2 - \frac{\pi}{12} K r^4 + \dots$$

where $A(r)$ is the area of disk of intrinsic radius r about p.

Example In $S^2, A(r) = 2\pi(1 - \cos r)$. The area is slightly smaller than expected when K is positive.

Global Theorems

Recall topological classification of *closed* (compact without boundary), *orientable* (abstract) surfaces:

Euler chracteristic χ

Theorem 1.7 *Let M be a closed surface. The Euler characteristic*

$$\chi(M) := \#\underbrace{faces}_{2\text{-}simplices} - \#\underbrace{edges}_{1\text{-}simplices} + \#\underbrace{vertices}_{0\text{-}simplices}$$

is independent of the choice of triangulation.

Definition n-simplex:$= \{x \in \mathbb{R}^n \,|x_1, \dots, x_n \geq 0, x_1 + \dots + x_n \leq 1\}$

Theorem 1.8 (Gauss-Bonnet Theorem)
Let (M, g) be a compact surface without boundary with Riemannian metric g. Then

$$\underbrace{\int_M K\, dA}_{\text{curvature integral quantitive, geometric}} = \underbrace{2\pi\chi(M)}_{\text{topological invariant, qualitative}} \in 2\pi\mathbb{Z}$$

Theorem 1.9 (Uniformization Theorem)
M compact surface without boundary. Then M possesses a metric g of constant Gauss curvature*:*

$$K \equiv \begin{cases} 1 & \text{iff } \chi > 0 & S^2 \\ 0 & \text{iff } \chi = 0 & T^2 \\ -1 & \text{iff } \chi < 0 & \text{surfaces with 2 or more holes} \end{cases}$$

Higher dimensions (preview)

(M^n, g) Riemannian manifold
g_p: inner product on each T_pM
How to define curvature without reference to extrinsic geometry?
Fact:
Given $p \in M, X \in T_pM$ there always exists a geodesic (locally length minimizing curve) with initial velocity $\frac{d\gamma}{dt}(0) = X$.
Fix $p \in M$.
Fix a 2-space $P \subseteq T_pM$. Let Q be the surface swept out by the geodesics γ_X with initial velocity X, where X ranges over unit vectors in P.
Define: $K(P) = K_p(P) :=$ Gauss curvature of Q at p (called *sectional curvature in planardirection P*)

$$K_p : \{\text{2-planes in } T_pM\} \to \mathbb{R}.$$

Clearly K_p is intrinsic.

Theorem 1.10 Cartan's Theorem*: If K is constant then M is locally isometric to either*

$$\begin{array}{lcll} S^n: & K & \equiv & c > 0 \\ \mathbb{R}^n: & K & \equiv & 0 \\ \mathbb{H}^n: & K & \equiv & -c < 0, \end{array}$$

where $\mathbb{H}^n$ is hyperbolic space.

Theorem 1.11 (Hadamard's Theorem) *If $K \leq -c < 0$ (and complete) then the universal cover of M is topologically equivalent to $\mathbb{R}^n$.*

Note If M is compact it follows that $\pi_1(M)$ is infinite.

Note
- Negative curvature makes geodesics spread out.
- Positive Curvature makes them come together (as in S^n, where they meet on the other side.)

Theorem 1.12 (Bonnet-Myers Theorem) *If $K \geq \beta > 0$, then M is compact with*

$$d_M(p, q) \leq \frac{\pi}{\sqrt{\beta}} \; \forall p, q \in M$$

This inequality is exact on S^2. Let p, q be antipodal points. We have

$$\begin{array}{rcl} K = & \frac{1}{R^2} & =: \beta \\ d(q, p) = & \pi R & = \dfrac{\pi}{\sqrt{\beta}} \end{array}$$

Note It follows that the universal cover M is also compact, so $|\pi_1(M)| < \infty$.

2 Differentiable Manifolds

- A topological manifold is a Hausdorff topological space such that each point has a neighborhood that is locally homeomorphic to $\mathbb{R}^n$
- A differentiable manifold is chatacterized by the additional condition that the overlap maps are smooth.

Definition let M be a set. A *chart* for M is a pair $(U, \psi), U \subseteq M, \psi : U \to \mathbb{R}^n$ injective, $\psi(U)$ open in $\mathbb{R}^n$.

$$\psi(p) = (x^1(p), \dots, x^n(p)) \quad \text{(coordinate functions on } U\text{)}$$

We call $\psi^{-1} : \psi(U) \subseteq \mathbb{R}^n \longrightarrow U \subseteq M$ a parametrization of U

$$\psi^{-1}(x_1, \dots x_n) = p$$

We cover M with charts:

$$M = \cup_{\alpha \in \mathcal{A}} U_\alpha$$

and examine their behaviour on an overlap

$$W := U_\alpha \cap U_\beta.$$

Definition We call (U_α, ψ_α) and (U_β, ψ_β) (*smoothly*) *compatible* if $\psi_\alpha(W), \psi_\beta(W)$ are open in $\mathbb{R}^n$ and the *overlap* (or *transition*) *map*

$$\psi_\beta \circ (\psi_\alpha^{-1}|_{\psi_\alpha(W)}) : \ \psi_\alpha(W) \to \psi_\beta(W)$$

and its inverse are infinitely differentiable.

Definition *A differentiable manifold of dimension* n is given by a set M equipped with a collection of charts $(U_\alpha, \psi_\alpha)_{\alpha \in \mathcal{A}}$ such that

i. $\cup_{\alpha \in \mathcal{A}} U_\alpha = M$

ii. each pair of charts is smoothly compatible

iii. the induced topology of M is Hausdorff

Motivation for ii.

$$\text{Let } f : M \to \mathbb{R}.$$

Then in coordinates:

$$f \circ \psi_\alpha^{-1} \text{ smooth} \Leftrightarrow f \circ \psi_\beta^{-1} \text{ smooth}$$

$$\underbrace{f \circ \psi_\alpha^{-1}}_{\text{on } \mathbb{R}^n} = \underbrace{(f \circ \psi_\beta^{-1})}_{\text{on } \mathbb{R}^n} \circ \underbrace{(\psi_\beta \circ \psi_\alpha^{-1})}_{\mathbb{R}^n \to \mathbb{R}^n}$$

Example

- $\mathbb{R}^n$

- any open set $M := U \subseteq \mathbb{R}^n$
  just one chart
$$\mathrm{id}_U : M \supseteq U \to U \subseteq \mathbb{R}^n$$

- graph of a smooth function
$$f : V \subseteq \mathbb{R}^n \to \mathbb{R} \ (V \text{ open})$$
  just one chart: projection from the graph to V via $(z, f(z)) \mapsto z$.

- any set $M \subseteq \mathbb{R}^n$ that can be written locally as a graph

- e.g.
$$S^n := \partial B_1 \subseteq \mathbb{R}^{n+1}$$
  needs $2(n+1)$ charts (of graph projection type)

- Möbius strip:
$$M := (0,3) \times (0,1)/\sim$$
  equivalence relation: $(x,y) \sim (x+2, y-1)$, $0 < x < 1,\ 0 < y < 1$.
  The natural projection is
$$\begin{aligned} \pi : (0,3) \times (0,1) &\to M \\ (x,y) &\to [(x,y)] := \text{ equivalence class of } (x,y) \end{aligned}$$
  2 charts:
$$\begin{aligned} \psi_1^{-1} := \pi|(0,2) \times (0,1) &\to M \\ \psi_2^{-1} := \pi|(1,3) \times (0,1) &\to M \end{aligned}$$

- $G(n,k) := \{\text{all } k\text{-dimensional subspaces of } \mathbb{R}^n\}$ This is called the *(real) Grassmannian* of k-planes in $\mathbb{R}^n$.
  Exercise What's its dimension?
$$\begin{aligned} \mathbb{R}P^n &:= \left\{\text{all lines through the origin in } \mathbb{R}^{n+1}\right\} \\ &= G(n+1, 1) \end{aligned}$$
  Exercise Find charts for $\mathbb{R}P^n$

- configuration space of all 3-4-5 triangles in $\mathbb{R}^2$

- configuration space of all (equilateral) 1-1-1 triangles

- Even the space of $\{a\text{-}a\text{-}a \text{ triangles in } \mathbb{R}^2 : a \geq 0\}$ is a manifold. **Exercise:** What manifold is this?

2.1 Topology of M

How to define a notion of open sets in M? We transfer them from $\mathbb{R}^n$ via charts. This results in a *local* test, as follows.

Definition $W \subseteq M$ is *open (in M)* if $\forall \alpha \in A, \psi_\alpha(W \cap U_\alpha)$ is open in $\mathbb{R}^n$.

Let $\mathcal{T} := \{\text{open sets } S \text{ in } M\}$

Proposition 2.1 (Exercise) *$\mathcal{T}$ has the following properties:*

i.
$$V, W \in \mathcal{T} \Rightarrow V \cap W \in \mathcal{T}$$

ii.
$$W_\beta \in \mathcal{T}\ \forall \beta \in B \Rightarrow \cup_{\beta \in B} W_\beta \in \mathcal{T}$$

iii.
$$\varnothing, M \in \mathcal{T}$$

A collection of subsets of a set M that satisfies (1)-(3) is called a *topology* on M, and $(M, \mathcal{T})$ is called a *topological space*.

Example The collection of open sets in a metric space (X, d) always satisfies (1)-(3). It is called the *topology induced by the metric d*.

In our case, M has *no metric*. $\mathcal{T}$ is called *the topology induced by the charts*. Using a topology one can express

- continuity
- convergence, topological boundaries
- paths
- connectedness
- simple connectedness, number of holes

Definition A map $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ between topological spaces is called a *homeomorphism* (or a *topological equivalence*, or *bicontinuous*) if f is bijective and preserves open sets:

$$U \in \mathcal{T} \Leftrightarrow f(U) \in \mathcal{S}.$$

Exercise Show that U_α is open in M, and each chart

$$\psi_\alpha : M \supseteq U_\alpha \to \psi_\alpha(U_\alpha) \subseteq \mathbb{R}^n$$

is a homeomorphism.

The topology on U_α is defined by $\mathcal{T}_{U_\alpha} := \{W \cap U_\alpha | W \in \mathcal{T})\}$ Verify: $\mathcal{T}_{U_\alpha}$ is a topology on U_α. It is called the *subspace topology* induced by $\mathcal{T}$ on U_α.

Definition $(X, \mathcal{T})$ is *Hausdorff* if any two points $x, y \in X, x \neq y$ can be separated by open sets, i.e. $\exists U, V$ in $\mathcal{T}$ so that $x \in U, y \in V, U \cap V = \varnothing$.

Observation: A metric space is Hausdorff.

Example

$$\mathcal{T} := \{\varnothing, \{a, b\}, \{b\}\}$$

(b converges to a but a doesn't converge to b)

Why Hausdorff?

Consider the example.

$$(x, 1) \sim (x, 2), x \neq 0$$
$$M := \mathbb{R} \times \{1\} \cup \mathbb{R} \times \{2\} / \sim$$

The 2 points at the origin cannot be separated by open sets! This space fulfills conditions (1)-(2) of definition of a smooth manifold (check!) but fails to be Hausdorff. This is highly undesirable: For example, M could never be given a metric.

2.1.1 Maximal Atlas

Suppose we have an *atlas*

$$\mathcal{A} = (U_\alpha, \psi_\alpha)_{\alpha \in A}$$

There may be *many* other charts (U, ϕ) that are compatible with each chart in $\mathcal{A}$. Let

$$\bar{\mathcal{A}} := \{\text{all charts } (U, \phi) \text{ compatible with each chart in } \mathcal{A}\}$$

Easy to verify: These charts are also compatible with each other. Thus $\bar{\mathcal{A}}$ is an atlas. $\bar{\mathcal{A}}$ is the (unique) maximal atlas containing $\mathcal{A}$.
We call $\bar{\mathcal{A}}$ the *differentiable structure* (or *smooth structure*) induced by $\mathcal{A}$.
We also observe that $\mathcal{T}_{\bar{\mathcal{A}}} = \mathcal{T}_{\mathcal{A}}$

Definition A *differentiable manifold (smooth manifold, C^∞ manifold)* is a pair $(M, \mathcal{A})$ where $\mathcal{A}$ is a maximal atlas (satisfies (1)-(3)).

Remark (Freedman/Donaldson 1980's)
Starting in $n = 4$, there are topological manifolds that cannot be given a smooth structure.

Smooth functions from $M \to N$

M^n, N^m smooth manifolds,

$$\phi : M \to N$$

a function.

Definition

i. ϕ is *smooth* if ϕ is smooth near each $p \in M$.

ii. ϕ is *smooth near* p if there *exist* charts ψ, χ

$$\begin{array}{rcl} p \in U & \stackrel{\psi}{\to} & \mathbb{R}^n \\ \phi(p) \in V & \stackrel{\chi}{\to} & \mathbb{R}^m \end{array}$$

such that $\phi(U) \subseteq V$

and

$$\chi \circ \phi \circ \psi^{-1} | \psi(U) : \psi(U) \to \mathbb{R}^m$$

is infinitely differentiable on U.

Remark Using the chain rule, it follows that ϕ is smooth in *all* charts.

Definition A function $f : (X, \mathcal{T}) \to (Y, \mathcal{S})$ is *continuous* provided

$$V \in \mathcal{S} \Rightarrow f^{-1}(V) \in \mathcal{T}$$

Proposition 2.2 *A smooth map between differentiable manifolds is* continuous *with respect to the topologies induced by the smooth structures.*

3 Tangents, differentials of maps

Tangent vectors

Here're two alternative ways of defining tangent vectors:

i. Identify together vectors in charts to equivalence classes via the equivalence relation $(X, \alpha, p) \sim (\tilde{X}, \beta, p)$ where

$$\tilde{X}^i = \sum_{j=1}^{n} \frac{\partial \left(\psi_\beta \circ \psi_a^{-1}\right)^i}{\partial x^j} X^j, \quad i = 1, \ldots, n.$$

ii. A tangent vector is a *directional derivative operator* coming from differentiation along some smooth curve.

3.1 Tangent vector as directional derivative operator

$$C^\infty(M) := \{\text{infinitely differentiable functions } M \to \mathbb{R}\}$$

Motivation

Let $X \in \mathbb{R}^n$ be a vector based at $p \in \mathbb{R}^n$. X yields a linear operator $C^\infty(\mathbb{R}^n) \to \mathbb{R}$ as follows: pick curve γ, $\gamma(0) = p$, $\dot{\gamma}(0) = X$, e.g. $t \mapsto p + tX$, then define

$$\begin{aligned} X : C^\infty(\mathbb{R}^n) &\to \mathbb{R} \\ f &\mapsto \left.\frac{d}{dt}\right|_0 f(\gamma(t)). \end{aligned}$$

Compute

$$\begin{aligned} X \cdot f &= \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(p) \, \frac{d\gamma^j}{dt}(0) \\ &= \sum_{j=1}^{n} \frac{\partial f}{\partial x^j}(p) \, X^j \end{aligned}$$

On a manifold, we have the curves γ but not yet X.

Definition Let $p \in M$. A *tangent vector to* M *at* p is a linear function

$$X : C^\infty(M) \to \mathbb{R}, \; f \mapsto X \cdot f$$

that arises as the directional derivative along some smooth curve starting at p, i.e.

$$\exists \gamma : (-\varepsilon, \varepsilon) \to M \text{ smooth}, \ \gamma(0) = p$$

such that

$$X \cdot f = \left.\frac{d}{dt}\right|_{t=0} f(\gamma(t)) \ \forall f \in C^\infty(M).$$

(One says that X is the *velocity vector* of γ at $t = 0$)

Definition

$$T_pM := \{(p, X) | \ X \text{ is a tangent vector to } M \text{ at } p\}$$

tangent space of M at p. Informally, we often use X to stand for the pair (X, p).

Expression in coordinates

i. *Coordinate vectors*
Let $p \in M$, $\psi : U \subseteq M \to \mathbb{R}^n$ a chart near p, $\tilde{p} := \psi(p)$. $\tilde{f} := f \circ \psi^{-1}$.
Consider the *coordinate curve*

$$\tilde{\beta}_i : t \mapsto \tilde{p} + te_i \text{ in } \mathbb{R}^n,$$

$$\beta_i := \psi^{-1} \circ \tilde{\beta}_i \quad \text{in } M.$$

Define

$$\left(\frac{\partial}{\partial x^i}\right)_p \equiv \left(\frac{\partial}{\partial x^i}\right)_{p,\psi} \in T_pM$$

by

$$\left(\frac{\partial}{\partial x^i}\right)_p \cdot f := \left.\frac{d}{dt}\right|_{t=0} f(\beta_i(t)).$$

Compute

$$\begin{aligned}
\left(\frac{\partial}{\partial x^i}\right)_p \cdot f &= \left.\frac{d}{dt}\right|_0 f \circ \beta_i \\
&= \left.\frac{d}{dt}\right|_0 \tilde{f} \circ \tilde{\beta}_i \\
&= \left.\frac{d}{dt}\right|_0 \tilde{f}(\tilde{p} + te_i) \\
&= \frac{\partial \tilde{f}}{\partial x^i}(\tilde{p})
\end{aligned}$$

Get $\left(\frac{\partial}{\partial x^1}\right)_p, \ldots, \left(\frac{\partial}{\partial x^n}\right)_p \in T_pM$, linearly independent in the vector space $\text{Hom}(C^\infty(M), \mathbb{R})$.

ii. **Claim** Any tangent vector X in T_pM is a linear combination of the $\left(\frac{\partial}{\partial x^i}\right)_p$'s.

Proof For some curve γ with $\gamma(0) = p$:

$$X \cdot f = \left.\frac{d}{dt}\right|_0 f(\gamma(t))$$

$$= \left.\frac{d}{dt}\right|_0 \underbrace{(f \circ \psi^{-1})}_{\tilde{f}(x_1,\ldots,x_n)} \circ \underbrace{(\psi \circ \gamma)}_{\tilde{\gamma}(t)}$$

$$= \sum_{j=1}^{n} \frac{\partial \tilde{f}}{\partial x^j}(\tilde{p}) \frac{d\tilde{\gamma}^j}{dt}(0)$$

with $\tilde{\gamma}(t) = (\tilde{\gamma}^1(t), \ldots, \tilde{\gamma}^n(t))$

$$= \left(\sum_{j=1}^{n} \frac{d\tilde{\gamma}^j}{dt}(0) \left(\frac{\partial}{\partial x^j}\right)_p \right) \cdot f$$

so

$$X = \sum_{j=1}^{n} \frac{d\tilde{\gamma}^j}{dt}(0) \left(\frac{\partial}{\partial x^j}\right)_p$$

Thus: T_pM is an n-dimensional vectorspace with basis $\left(\frac{\partial}{\partial x^1}\right)_p, \ldots \left(\frac{\partial}{\partial x^n}\right)_p$

□

iii. Consider the following possible alternative definition of a tangent vector: A tangent vector to M at p is a linear functional

$$X : C^\infty(M) \to \mathbb{R}$$

that satisfies the Leibniz rule:

$$X \cdot (fg) = (X \cdot f)g(p) + f(p)X \cdot g$$

Exercise Prove this for $n = 1$, and find out if it's true for general n.

3.2 Differential of a map

Let $\phi : M^n \to N^m$ be smooth, $p \in M$.

Definition Define $d\phi(p) \equiv d\phi_p : T_pM \to T_{\phi(p)}N$ as follows: Let $X \in T_pM$, choose a path α such that $X =$ velocity vector of α at $t = 0$, i.e.

$$X \cdot f = \left.\frac{d}{dt}\right|_0 f(\alpha(t)) \ \forall f \in C^\infty(M),$$

Let $\beta = \phi \circ \alpha$. Define $(Y \equiv) d\phi(p)(X) :=$ velocity vector of β at $t = 0$ i.e.

$$Y \cdot g := \left.\frac{d}{dt}\right|_0 g(\beta(t)) \ \forall g \in C^\infty(N).$$

Since $\beta(0) = \phi(\alpha(0)) = \phi(p)$, we get $Y \in T_{\phi(p)}N$.

Observe:

$$\begin{aligned} Y \cdot g &= \left.\frac{d}{dt}\right|_0 g(\phi(\alpha(t))) \\ &= \left.\frac{d}{dt}\right|_0 (g \circ \phi)(\alpha(t)) \\ &= X \cdot (g \circ \phi) \end{aligned}$$

which shows that Y depends only on X and not on the choice of α. This also shows that $d\phi(p)$ is linear. (We could have taken $Y \cdot g := X(g \circ \phi)$ to be the definition of $d\phi_p(X)$)

In coordinates

Let $X \in T_pM, \quad Y := d\phi(p)(X) \in T_qM, \quad q := \phi(p)$.
Write

$$X = X^i \left(\frac{\partial}{\partial x^i}\right)_p, \quad Y = \underbrace{Y^j \left(\frac{\partial}{\partial y^j}\right)_q}_{\sum_{j=1}^m}$$

Einstein summation convention: paired indices, one upper, one lower, are summed over appropriately.
We want to express

$$Y^j = ? \cdot X^i.$$

Set $\tilde{\phi} := \chi \circ \phi \circ \psi^{-1}, \ \tilde{g} := g \circ \chi^{-1}$
Compute:

$$\begin{aligned}
Y \cdot g &= X \cdot (g \circ \phi) \\
&= X^i \left(\frac{\partial}{\partial x^i}\right)_p \cdot (g \circ \phi) \\
&= X^i \left(\frac{\partial}{\partial x^i}\right)_p \cdot \left[\underbrace{(g \circ \chi^{-1})}_{\tilde{g}} \circ \underbrace{(\chi \circ \phi \circ \psi^{-1})}_{\tilde{\phi}} \circ \psi\right] \\
&= X^i \left(\frac{\partial}{\partial x^i}\right)_p \tilde{g} \circ \tilde{\phi} \circ \psi \\
&= X^i \frac{\partial(\tilde{g} \circ \tilde{\phi})}{\partial x^i}(\tilde{p}) \quad {}^{1} \\
&= X^i \frac{\partial \tilde{g}}{\partial y^j}(\tilde{q}) \frac{\partial y^j}{\partial x^i}(\tilde{p}) \quad \text{(chain rule)} \\
&= \left(X^i \frac{\partial y^j}{\partial x^i}(\tilde{p}) \left(\frac{\partial}{\partial y^j}\right)_q\right) \cdot g
\end{aligned}$$

i.e.

$$Y = X^i \frac{\partial y^j}{\partial x^i}(\tilde{p}) \left(\frac{\partial}{\partial y^j}\right)_q$$

i.e.

$$Y = Y^j \left(\frac{\partial}{\partial y^j}\right)_q,$$

where

$$\underbrace{Y^j}_{m} = \underbrace{\frac{\partial y^j}{\partial x^i}(\tilde{p})}_{m \times n} \underbrace{X^i}_{n}$$

Shows: $d\phi(p)$ is given in coords by the matrix

$$\frac{\partial y^j}{\partial x^i} \left(\equiv \frac{\partial \tilde{\phi}^j}{\partial x^i}\right)$$

Proposition 3.1 *(Chain rule)*

[1] previously showed: $\left(\frac{\partial}{\partial x^i} \cdot f = \frac{\partial \tilde{f}}{\partial x^i}(\tilde{p}), \ \tilde{f} = f \circ \psi^{-1}\right)$

If

$$M \xrightarrow{f} N \xrightarrow{g} P$$

$$T_pM \xrightarrow{df_p} T_{f(p)}N \xrightarrow{dg_{f(p)}} T_{g(f(p))}P$$

then:

$$d(g \circ f)_p = dg_{f(p)} \circ df_p.$$

Proof Transfer the chain rule

$$\mathbb{R}^m \to \mathbb{R}^n \to \mathbb{R}^p$$

to M, N, P via charts.

□

Products

Let M^m, N^n : be smooth manifolds with atlases

$$\begin{aligned} \mathcal{A} &= (U_\alpha, \psi_\alpha)_{\alpha \in A} \\ \mathcal{B} &= (V_\beta, \chi_\beta)_{\beta \in B} \end{aligned}$$

where

$$\begin{aligned} \psi_\alpha : U_\alpha &\to \mathbb{R}^m \\ \chi_\beta : V_\beta &\to \mathbb{R}^n. \end{aligned}$$

Give $M \times N$ the charts

$$\begin{aligned} \psi_\alpha \times \chi_\beta : U_\alpha \times V_\beta &\to \mathbb{R}^m \times \mathbb{R}^n, \\ (p, q) &\mapsto (\psi_\alpha(p), \chi_\beta(q)) \end{aligned}$$

and the atlas

$$\mathcal{A} \times \mathcal{B} := \{(U_\alpha \times V_\beta, \psi_\alpha \times \chi_\beta) \,|\alpha \in A, \beta \in B\}$$

Canonical projections:

$$\begin{aligned} \pi_M : \quad M \times N &\to M \\ (p, q) &\mapsto p \\ \pi_N : \quad M \times N &\to N \\ (p, q) &\mapsto q \end{aligned}$$

Proposition 3.2 ***(Exercise)***
Show $(M \times N, \mathcal{A} \times \mathcal{B})$ *yields a manifold, and* π_M*,* π_N *are smooth.*

Example $\mathbb{R}^p \times \mathbb{R}^q$ is the same as $\mathbb{R}^{p+q}$

$$S^1 \times S^1 = T^2 \quad \text{(2-Torus)}$$

$$T^n := S^1 \times \cdots \times S^1 \quad (n\text{-torus})$$

Example $\Xi :=$ {space of right handed 3-4-5 triangles in $\mathbb{R}^2$}
Project $T \in \Xi$ to $p(T) \in \mathbb{R}^2$ (the sharpest vertex) and to $\Theta(T) \in S^1$ (the angle that the length 4 side, directed away from $p(T)$, makes with the positive x-axis). Then the bijection $(p, \Theta) : \Xi \to \mathbb{R}^2 \times S^1$ shows $\Xi = \mathbb{R}^2 \times S^1$.

Tangent bundle

M smooth. Define

i.
$$T_pM := \{(p, X)|\ X \in \operatorname{Hom}(C^\infty(M), \mathbb{R}) \text{ is a tangentvector to } M \text{ at } p\}$$

so $0_p \neq 0_q$ when $p \neq q$. $(p, X) \equiv X$ (abuse of notation)

ii.
$$TM := \bigcup_{p\in M} T_pM = \{(p, X) : p \in M, X \in T_pM\}$$

T_pM is called the *fiber* at p.

iii.

$$\begin{aligned} \pi : \quad TM &\to M \\ (p, X) &\mapsto p \end{aligned}$$

(*canonical projection*)

Proposition 3.3 *TM has the structure of a $2n$-dimensional manifold.*

Let (U, ψ) be a chart for M

$$p \in U \subseteq M \quad \overset{\psi}{\mapsto} \quad \psi(p) = \big(x^1(p), \ldots, x^n(p)\big) \in \mathbb{R}^n$$

$$X^i \left(\frac{\partial}{\partial x^i}\right)_p = X \in T_pM \quad \overset{d\psi(p)}{\to} \quad \big(X^1, \ldots, X^n\big) \in \mathbb{R}^n. \qquad \text{(check this!)}$$

Define a chart for TM as follows:
Set

$$\mathrm{U} := TU = \pi^{-1}(U) = \cup_{p\in U} T_pM \subseteq TM$$

Define

$$\begin{aligned}
\Psi : \mathrm{U} &\to \psi(U) \times \mathbb{R}^n \text{ by} \\
(p, X) &\mapsto \big(x^1(p)), \ldots, x^n(p)), X^1, \ldots, X^n\big) \\
&= \Bigg(\underbrace{x^1, \ldots, x^n}_{\text{coords of } p}, \quad \underbrace{X^1, \ldots, X^n}_{\text{coords of } X \text{ within } T_pX}\Bigg)
\end{aligned}$$

The associated parametrization has a some what simpler form:

$$\Psi^{-1} : \big(x^1, \ldots, x^n, X^1, \ldots, X^n\big) \mapsto \left(\underbrace{\psi^{-1}(x^1, \ldots, x^n)}_{p}, \sum X^i \left(\frac{\partial}{\partial x^i}\right)_p\right)$$

Exercise The charts (U, Ψ) are compatible and give TM the structure of a $2n$-manifold. $\pi : TM \to M$ smooth. TM is *locally* a product $\psi(U) \times \mathbb{R}^n$

Example S^1
Coordinates:

$$\begin{aligned}
\mathbb{R} &\to S^1 \\
\theta &\mapsto [\theta] := \theta + 2\pi k,\ k \in \mathbb{Z}
\end{aligned}$$

$$\begin{array}{ccc}
TS^1 & \ni \left([\theta], a\left(\frac{\partial}{\partial\theta}\right)_{[\theta]}\right) & [\theta] \in S^1, a \in \mathbb{R} \\
\cong \big\downarrow \text{ preserves smooth structure} & \big\downarrow & \\
S^1 \times \mathbb{R} & \ni ([\theta], a) &
\end{array}$$

$TS^1 \simeq S^1 \times \mathbb{R}$ cylinder, a product, of the *base* S^1 with $\mathbb{R}$.

$$\begin{aligned}
TS^2 &\ncong S^2 \times \mathbb{R}^2 \\
TS^3 &\cong S^3 \times \mathbb{R}^3 \\
TS^4 &\ncong S^4 \times \mathbb{R}^4 \\
&\vdots
\end{aligned}$$

Definition A *smooth vector field* on M is a smooth function $X : M \to TM$ such that $X(p) \in T_pM \ \forall p \in M$.
In coordinates $p \xrightarrow{\psi} (x^1, \dots, x^n)$

$$\begin{aligned} X(x^1, \dots x^n) &= (x^1, \dots, x^n, X^1(x^1, \dots, x^n), \dots, X^n(x^1, \dots, x^n)) \\ &\overset{\text{abuse}}{=} (X^1(x^1, \dots, x^n), \dots, X^n(x^1, \dots, x^n)) \end{aligned}$$

Evidently, X is a smooth vector field $\Leftrightarrow$ components $X^1(x^1, \dots, x^n), \dots, X^n(x^1, \dots, x^n)$ of X are smooth.
Semi intrinsically, we write

$$X(p) = \sum_{i=1}^{n} \underbrace{X^i\left(x^1, \dots, x^n\right)}_{C^\infty} \left(\frac{\partial}{\partial x^i}\right)_p$$

Question: How many pointwise linearly independant vector fields can we find on S^n? Specifically, we require $\forall p \in S^n$, $e_1(p), \dots e_k(p)$ are linearly independent in T_pS^n.

Theorem 3.4 *There is no nowhere-vanishing vector field on S^2.*

Theorem 3.5 *(F.Adams) Gives a peculiar formula for the maximum number of pointwise linear independent vectorfields on S^n. (See Greenberg & Harper.)*

$TS^1 \cong S^1 \times \mathbb{R}$	S^1	1
	S^2	0
$TS^3 \cong S^3 \times \mathbb{R}^3$	S^3	3
	S^4	0
	S^5	$\neq 0, 5$
	S^6	0
$TS^7 \cong S^7 \times \mathbb{R}^7$	S^7	7

4 Submanifolds, diffeomorphisms, immersions and submersions

Reference: Guillemin and Pollack Chap 1, pp 1-27
Let M be a smooth manifold, $N \subseteq M$ a subset.

Definition N is a (*smooth*) *k-dimensional submanifold* of M if $\forall x \in N$, $\exists U \ni x$ open and a chart $\psi : U \to \mathbb{R}^n$ such that

$$\psi(N \cap U) = (\mathbb{R}^k \times \{0\}) \cap \psi(U).$$

Atlas for N:

$\mathcal{A}_N := \big\{(V, \chi) | \quad V := N \cap U \quad \chi := \psi | N \cap U : N \cap U \to \mathbb{R}^k, (U, \psi) \text{ as above}\big\}$.

Examples

- open subset of a manifold
- S^n in $\mathbb{R}^{n+1}$
- S^{n-1} in S^n
- (prove later) classical groups $O(n), U(n), Sp(n), \ldots$ are submanifolds of $M^{n \times n} \cong \mathbb{R}^{n^2}$
- open upper hemisphere of S^n, in $\mathbb{R}^{n+1}$

Proposition 4.1

- $(N, \mathcal{A}_N)$ *is a smooth k-manifold.*
- The inclusion map *of N in M $i \equiv i_{N \subseteq M}$:*

$$\begin{array}{ccc} N & \to & M \\ p & \mapsto & p \end{array}$$

  is smooth.

- *It's differntial*

$$di_p : T_pN \to T_pM$$

  is an injection $\forall p$, modelled on the linear inclusion $\mathbb{R}^k \subseteq \mathbb{R}^n$.

- *The subspace topology on N coincides with the chart topology. For any $N \subseteq (M, \mathcal{T}_M)$ (not necessarily a submanifold), we define $\mathcal{T}_N := \{U \cap N | U \in \mathcal{T}_M\}$. called the* subspace topology induced on N from *$(M, \mathcal{T}_M)$*

  Proposition 4.2 *$\mathcal{T}_N$ is a topology on N*

Big Questions:

i. When is the image of a smooth map a submanifold?

ii. When is the zero-set of a smooth map a submanifold?

4.1 Immersions, submersions, diffeomorphisms

Let

$$\begin{array}{rccc} f: & M^n & \to & N^m \\ df_p: & T_pM & \to & T_{f(p)}N. \end{array}$$

be smooth, and consider

Definition

i. f is an *immersion* if df_p is injective $\forall p \in M$

ii. f is a *submersion* if df_p is surjective $\forall p \in M$

iii. f is a *diffeomorphism* if f is bijective and f^{-1} is also smooth. (NB: then $f^{-1} \circ f = id_M$, $(df^{-1})_{f(p)} \circ df_p = id_{T_pM}$, so df_p is an isomorphism)

Correspondingly, we have

i. Local immersion theorem (Blatter II p.106)

ii. Local submersion theorem ($\equiv$ Implicit function theorem) (Blatter II p.99)

iii. Inverse function theorem (Blatter II p.88)

The first two are dual and both are proved from iii.

Diffeomorphisms

$$(M, \mathcal{A}) \underset{f^{-1}}{\overset{f}{\rightleftarrows}} (N, \mathcal{B})$$

f diffeomorphism $\Leftrightarrow$ f^{-1} diffeomorphism.
Write: $M \overset{\text{diff}}{\cong} N$
It means: M and N "look the same" from a differentiable viewpoint.

Advanced Fact (Taubes/Donaldson 80's)

Starting in $n = 4$, a topological manifold can have 0,1 or ≥ 2 distinct (i.e. non-diffeomorphic) differentiable structures.

Example (Milnor 50's) The topological manifold S^7 has 28 distinct differentiable structures.
Standard one: $S^7 := \{x \in \mathbb{R}^8 | \, |x| = 1\}$

Theorem 4.3 (Inverse function theorem) *Let $f : M \to N$ be smooth. If $df_p : T_pM \to T_{f(p)}N$ is an isomorphism, then f is a diffeomorphism near p, that is, $\exists U \ni p, V \ni f(p)$ open such that $f|U : U \to V$ is a diffeomorphism.*

Proof Transfer the usual Inverse Function Theorem from $\mathbb{R}^n$ to M, N via charts.

$\square$

Definition Let $f : M \to N$

i. f is a *local diffeomorphism* if every $p \in M$ has a neighborhood $U \ni p$ such that $f(U)$ is open in N and $f|U : U \to f(U)$ is a diffeomorphism.

ii. f is a (*smooth*) *covering map* if every $q \in N$ has a neighborhood $V \ni q$ such that $f^{-1}(V) = \cup_{\delta \in \Delta} U_\delta$, where the U_δ are open disjoint sets in M, and $f|U_\delta : U_\delta \to V$ is a diffeomorphism for each δ.

Clear:

Covering map $\overset{\Rightarrow}{\not\Leftarrow}$ local diffeomorphism

Exercise Prove that the number of preimage points $f^{-1}(q)$ is constant on each *connected component* of N, if f is a covering map.

Example

$$\begin{aligned} S^n &\overset{\pi}{\to} \mathbb{R}P^n \\ p &\mapsto \pi(p) := \text{line through } p \text{ and } 0 \end{aligned}$$

π is a covering map (where we give $\mathbb{R}P^n$ a suitable smooth structure). Each $L \in \mathbb{R}P^n$ has two preimage points $p, -p$ in S^n.

Let Γ be a group of diffeomorphisms from M to M, i.e.

$$\begin{array}{rcl} id_M \in \Gamma, \quad g \in \Gamma & \Rightarrow & g^{-1} \in \Gamma \\ g, h \in \Gamma & \Rightarrow & g \circ h \in \Gamma \end{array}$$

Definition Γ acts *freely and properly discontinuously* on M if $\forall p \in M\ \exists U_{\text{open}} \ni p$ such that

$$g \neq h \in \Gamma \Rightarrow g(U) \cap h(U) = \varnothing.$$

Example

$$\mathbb{Z}_2 \cong \{id_{s^n}, g\}$$

where $g(x) := -x,\ g^2 = id_M$. Then $\mathbb{Z}_2$ acts freely and properly discontinuous on S^n.

Definition Let Γ be a group and M a manifold. Γ *acts smoothly on* M if there is a homomorphism of Γ to the group of diffeomorphisms ($\equiv \mathrm{Diff}(M)$) of M.

Example $\mathbb{Z}^n$ acts freely and properly discontinuously on $\mathbb{R}^n$ by translation.

Notation

$$\begin{array}{rcll} \rho : \Gamma & \to & \mathrm{Diff}(M) & \text{group action} \\ g & \mapsto & \rho(g) & \\ \rho(g)(x) & \equiv & g(x) & \end{array}$$

Definition We call $\Gamma \cdot x := \{g(x) | g \in \Gamma\}$ *the orbit of* x *under action of* Γ.

M decomposes into a disjoint union of orbits. Specifically one can easily see:

i. for all $x, y \in M$, either $\Gamma \cdot x = \Gamma \cdot y$ or $\Gamma \cdot x \cap \Gamma \cdot y = \varnothing$

ii. $M = \cup_{x \in M} \Gamma \cdot x$

Each orbit is an equivalence class for the relation

$$x \sim y \Leftrightarrow y = g(x)\ \exists g \in \Gamma.$$

We obtain:

$$\begin{array}{rcl} \pi : \quad M & \to & M/\Gamma \\ x & \mapsto & \Gamma \cdot x \end{array}$$

$$\begin{array}{rcl} M/\Gamma & := & \{\text{set of orbits}\} \\ & = & \{\Gamma \cdot x | x \in M\} \\ & = & M/\sim \end{array}$$

Theorem 4.4 *(Exercise)*
If Γ acts freely and properly discontinuously on M, then $\pi : M \to M/\Gamma$ induces a smooth structure on M/Γ such that π is a covering map.

Warning Not every covering map comes from an appropriate group action!

Exercise Find an example.

Definition A subset A of a topological space X is *discrete* if for each $x \in A$ $\exists U$ open such that $A \cap U = \{x\}$.

Exercise G Lie group (a manifold such that the group operations are smooth), Γ discrete subgroup (not necessarily normal!) and G/Γ coset space of Γ in G

- $SL(2,\mathbb{R})/SL(2,\mathbb{Z}) =?$ (3-manifold)
- $S^3/\{\pm 1\} \cong \mathbb{R}P^3$, $S^3/\mathbb{Z}_\ell$ (some 3-manifold)

$$\mathbb{Z}_\ell := \left\{ e^{2\pi i k/\ell} | k = 0, \ldots, \ell - 1 \right\}$$

Exercise

Find all the manifolds (up to diffeomorphism) of the form $\mathbb{R}^2/\Gamma$, Γ acts freely and properly discontinuously on $\mathbb{R}^2$ *by isometries* (translations, rotations, refections and slide reflections).

* Same problem for $\mathbb{R}^3$.

4.2 Immersions

An *immersion* is a function such that

$$\begin{array}{rccll} f: & M^k & \to & N^n & \text{smooth} \\ df(p): & T_pM & \to & T_{f(p)}N & \textit{is an injection.} \end{array} \quad (\Rightarrow k \leq n)$$

Example The inclusion map $i : M \to N, x \mapsto x$ of any submanifold M of N is an immersion.

Example (curves) A regular curve ($\dot{\gamma}(t) \neq 0$)

$$\mathbb{R} \ni t \mapsto \gamma(t) \in \mathbb{R}^2$$

is an immersion.

Example (Canonical linear immersion)

$$\begin{aligned} i : \mathbb{R}^k &\to \mathbb{R}^n \\ (x^1, \ldots, x^k) &\mapsto (x^1, \ldots, x^k, 0, \ldots, 0) \end{aligned}$$

Theorem 4.5 (Local Immersion Theorem) *Let $f : M \to N$ be smooth, $p \in M$ be fixed. Suppose*

$$df_p : T_pM \to T_{f(p)}N$$

is injective. Then there exist local coordinates $(x^1, \ldots, x^k)$ about p, $(y^1, \ldots, y^n)$ about $f(p)$ such that in these coordinates, f has the form

$$(x^1, \ldots, x^k) \mapsto (x^1, \ldots x^k, 0, \ldots, 0) = (y^1, \ldots, y^n)$$

near p.
This says "f is smoothly equivalent *to i". This means that any immersion can be straightend, out at least locally.*
Proof later.

Corollary 4.6 *If df_p is injective at p then df_p will be injective for all q near p.*
So $\{p \in M | df_p$ injective$\}$ is open. *"That is ,injectivity of the differential of f is an open condition on points of M".*

Corollary 4.7 *The image under an immersion of a sufficiently small open set of M is a* submanifold *of N.*

Question:

When is the image of a smooth map a *submanifold* of the target manifold?

Theorem 4.8 *If $f : M \to N$ is an injective immersion and a homeomorphism onto it's image[2], then $f(M)$ is a smooth submanifold of N and f is a diffeomorhism from M to $f(M)$.*

Proof

[2]This means: $f : M \to f(M)$ is a homeomorphism (where $f(M)$ has the subspace topology coming form N).

i. Fix $q \in f(M), p := f^{-1}(q)$ (unique, $f : M \to f(M)$ bijective). By the Local Immersion Theorem, $\exists U_{\text{open}} \ni p$, $W_{\text{open}} \ni q$ such that

$$f|U : U \to W$$

is the cannonical linear immersion

$$i : \mathbb{R}^k \to \mathbb{R}^k \times \mathbb{R}^{n-k}$$

in coordinate systems $(x^1, \ldots, x^k)$ on U and $(y^1, \ldots, y^n)$ on W. Thus $f(U)$ is a submanifold of N and $f|U$ is a diffeomorphism from U to $f(U)$. Since f is a homeomorphism from M to $f(M)$ and U is open in M, $f(U)$ is open in $f(M)$, i.e.

$$f(U) = V \cap f(M)$$

for some V open in N.

This tells us: $f(U)$ is *cleanly separated via V from the rest of $f(M)$.*

In fact, we have that $f(M) \cap V$ is a submanifold of N.(Recall that in the coordinates $y^1, \ldots, y^n$ on N near q, $f(M)$ maps to an open set in $\mathbb{R}^k$)

Since such a V can be found about any point q of $f(M)$, it follows that $f(M)$ is a submanifold of N.

ii. $f : M \to f(M)$ is a local diffeomorphism by the above, and $f : M \to f(M)$ is a homeomorphism. So $f^{-1} : f(M) \to M$ exists. Using the Inverse Function Theorem, f^{-1} is smooth.

□

Homeomorphism-ness is hard to test directly.

Definition If $f : M \to N$ satisfies the conclusions of the previous Theorem (ie $f(M)$ is a submanifold of N and $f : M \to f(M)$ is a diffeomorphism), we call f an *embedding* of M in N.

Theorem 4.9 *Suppose $f : M \to N$ is an injective immersion and M is* compact*. Then f is an embedding.*

Proof Must show: $f : M \to f(M)$ homeomorphism. Note that f is bijective and continuous. Thus it suffices to show that f^{-1} is continuous, i.e. show: if U open in M then $f(U)$ is open in $f(M)$.

$$\begin{array}{rcl} U \text{ open in } M & \Rightarrow & M \setminus U \text{ closed in } M \\ & \Rightarrow & M \setminus U \text{ compact (since } M \text{ is compact} \\ & \Rightarrow & f(M \setminus U) = f(M) \setminus f(U) \text{ compact} \\ & \Rightarrow & f(M) \setminus f(U) \text{ closed in } f(M) \\ & \Rightarrow & f(U) \text{ open in } f(M). \end{array}$$

□

Proof (*Local Immersion Theorem*)
The theorem is entirely local, so without loss of generality we may assume

$$f : \mathbb{R}^k \supseteq U \to V \subseteq \mathbb{R}^n, \; U, V \text{ open}, \; p = 0$$

Without loss of generality (via postcomposition with a *linear* tronsformation of $\mathbb{R}^n$) we may assume

$$\begin{array}{rcl} df_p = i : \mathbb{R}^k & \to & \mathbb{R}^n \\ (x^1, \ldots, x^k) & \mapsto & (x^1, \ldots, x^k, 0, \ldots, 0) \\ & & \text{(canonical linear immersion)} \end{array}$$

To apply the Inverse Function Theorem we *augment* $\mathbb{R}^k$ to $\mathbb{R}^n$ by adding $n - k$ new variables. We extend f to a new function F by

$$\begin{array}{rcl} U \times \mathbb{R}^{n-k} & \to & \mathbb{R}^k \times \mathbb{R}^{n-k} \\ (x', x'') & \mapsto & f(x') + (0, x'') \end{array}$$

Compute for: $(X', X'') = X \in T_P(U \times \mathbb{R}^{n-k}) = \mathbb{R}^k \times \mathbb{R}^{n-k}$

$$\begin{array}{rcl} dF_p(X', X'') & = & \underbrace{df_p}_{i}(X') + (0, X'') \\ & = & (X', 0) + (0, X'') \\ & = & (X', X'') \end{array}$$

i.e.

$$dF_p = \mathrm{id}_{\mathbb{R}^n}$$

As matrices:

$$dF_p = \left(\begin{array}{c|c} \underbrace{df_p}_{x'} & \underbrace{\begin{array}{c} 0 \\ \hline I \end{array}}_{x''} \end{array} \right) \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = I$$

By the Inverse Function Theorem, $\exists W$ open $\ni p$, $F(W)$ open $\ni F(p, 0) = f(p)$ such that

$$F|W : W \to F(W)$$

is a diffeomorphism. So $G := (F|W)^{-1}$ is a valid chart for $F(W)$. So we can use $(x^1, \ldots, x^n)$ as coordinates on $F(W)$. Let $U_1 := W \cap (U \times \{0\})$.

Get: $(x^1, \ldots, x^k)$ coordinates on U,
$(X^1, \ldots, X^n)$ coordinates on $F(W)$

Then in these coordinates f has the form

$$(x^1, \ldots x^k) \mapsto (x^1, \ldots, x^k, 0, \ldots, 0).$$

□

Theorem 4.10 (Graphical Image Theorem) *(Restatement of Local Immersion Theorem)*
The image of a smooth map whose differential is injective at one point can be written locally, in the original *target varibles $(y^1, \ldots, y^n)$, as the graph of $(n - k)$ of the variables as a function of remaining k.*

Recall that if $f : M \to N$ is injective immersion and M compact then f is an embedding. Let's try to generalize this to M noncompact.

Definition $f : X \to Y$ is *proper* if $K \subseteq Y$, K compact $\Rightarrow f^{-1}(K)$ compact

Theorem 4.11 *If $f : M \to N$ injective immersion and* $\boxed{\textit{proper}}$ *then f is an embedding.*

Proof Exercise.

□

Example $\mathbb{R} \to T^2$ with an irrational slope: injective immersion, not proper. The image is dense in T^2 so it isn't an embedding.

Definition We call a topological space $(X, \mathcal{T})$ *second countable* if there exists a countable collection of open sets that generate the topology $\mathcal{T}$ via arbitrary unions, i.e. $\mathcal{T}$ has a *countable base.*

Example

$\mathbb{R}$	$\left\{\left(\frac{p}{q}, \frac{r}{s}\right) \lvert p, q, r, s \in \mathbb{Z},\ q, s \neq 0\right\}$	countable base
$\mathbb{R}^n$	products of such intervals:	countable base

Theorem 4.12 (Whitney Theorem) *Every (paracompact or second countable) smooth n-manifold can be embedded smoothly in $\mathbb{R}^{2n}$.*

Example

$S^1 \subseteq \mathbb{R}^2$	embedding
$\mathbb{R}P^2 \subseteq \mathbb{R}^4$	Veronese embedding
$\mathbb{R}P^2 \to \mathbb{R}^3$	Boy's immersion

There exist no embedding of $\mathbb{R}P^2$ in $\mathbb{R}^3$

4.3 Submersions

Zero Sets

Question $f : M \to N$ smooth. When is $f^{-1}(q)$ a submanifold of M?

Example

$$f : \mathbb{R}^2 \to \mathbb{R}$$

$f(x, y) := x^3 - y^2$, $f^{-1}(0)$ is a cone with a cusp (not smooth at $(0, 0)$

$$\nabla f = (3x^2, 2y)$$

Consider

$$\begin{aligned} f : M &\to N \text{ smooth} \\ df_p : T_pM &\to T_{f(p)}N \end{aligned}$$

We require: df_p *surjective* $\forall p \in M$.

Example (Canonical linear projection) Let $n \geq k$ and define

$$\begin{aligned} \pi : \mathbb{R}^n &\to \mathbb{R}^k \\ (x^1, \ldots, x^n) &\mapsto (x^1, \ldots, x^k). \end{aligned}$$

Then π is a submersion.

Example

$$\begin{array}{ccc} & M \times N & \\ {}^{\pi_M}\swarrow & & \searrow^{\pi_N} \\ M & & N \end{array}$$

Then π_M, π_N are submersions.

Example (Exercise) $TM \xrightarrow{\pi} M$ is a submersion.

Theorem 4.13 (Local Submersion Theorem) *$f : M^n \to N^k$ smooth, $p \in M$, $df_p : T_pM \to T_{f(p)}N$ surjective. Then there are coordinates $(x^1, \ldots, x^n)$ near p, $(y^1, \ldots, y^k)$ near $f(p)$, such that f has the form*

$$(x^1, \ldots, x^n) \mapsto (y^1, \ldots, y^k)$$

Notation:

$$\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \ni (x^1, \ldots, x^k, x^{k+1}, \ldots, x^n) = (x', x'')$$

$$\begin{array}{rll} \pi' : & \mathbb{R} \to \mathbb{R}^k, & x \mapsto x' \\ \pi'' : & \mathbb{R}^n \to \mathbb{R}^{n-k}, & x \mapsto x'' \end{array}$$

Proof Since the theorem is local, we may work in open sets in Euclidean space:

$$\begin{array}{ccc} f : U \subseteq \mathbb{R}^n & \to & V \subseteq \mathbb{R}^k \\ (x^1, \ldots, x^n) & & (y^1, \ldots, y^k) \end{array}$$

U, V open.
Precomposing f with an appropriate linear transformation $\mathbb{R}^n \to \mathbb{R}^n$, we may assume

$$\begin{array}{rcl} df_p = \pi' : \mathbb{R}^n & \to & \mathbb{R}^k \\ (x', x'') & \mapsto & x' \end{array}$$

To apply the Inverse Function Theorem, *complete* f to a map F as follows:

$$\begin{array}{rcl} F : U & \to & V \times \mathbb{R}^{n-k} \\ (x', x'') & \mapsto & (f(x', x''), \underbrace{\pi''(x)}_{\equiv x''}) \end{array}$$

Now let $X = (X', X'') \in T_p(\mathbb{R}^k \times \mathbb{R}^{n-k}) = \mathbb{R}^k \times \mathbb{R}^{n-k}$

Compute

$$\begin{aligned} dF_p(X', X'') &= \Big(\underbrace{df_p}_{\pi'}(X', X''), \underbrace{d\pi''_p}_{\pi''}(X', X'')\Big) \\ &= (X', X''). \end{aligned}$$

So $dF_p = \mathrm{id}_{\mathbb{R}^n}$ is an isomorphism.

$$\left(dF_p = \left(\begin{array}{c|c} \underbrace{df_p}_{x'} & \underbrace{\begin{array}{c} 0 \\ \hline I \end{array}}_{x''} \end{array} \right) \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = I \right)$$

Thus by the Inverse Function Theorem, $\exists U_1 \subseteq U$ open, $W \subseteq V \times \mathbb{R}^{n-k}$ open such that

$$U_1 \xrightarrow{F|U_1} W$$

is a diffeomorphism. So $F|U_1$ is a valid chart map and we may replace the coordinates $x^1, \ldots, x^n$ on U_1 by the coordinates $y^1, \ldots y^n$ coming form W. Then U_1 has the coordinates $(y^1, \ldots, y^n)$. $V \cap (W \cap \mathbb{R}^k \times \{0\})$ has coordinates $(y^1, \ldots, y^k)$. In these coordinates, f is represented by

$$(y^1, \ldots, y^n) \mapsto (y^1, \ldots, y^k).$$

□

Corollary 4.14 *df_p surjective at $p \Rightarrow df_p$ surjective for all q near p (i.e. surjectivity of df is an open condition in the domain manifold.)*

We return to our question:

When is the preimage $f^{-1}(q)$ a submanifold of M?

Corollary 4.15 *Let $f : M^n \to N^k$ be a submerison. Then $f^{-1}(q)$ is an $(n-k)$-dimensional submanifold of M for any $q \in N$.*

Note that the Local Submersion Theorem is really the Implicit Function Theorem in disguise.
We can be more precise in an answer to the above question.

Definition $f : M \to N$ smooth

- $p \in M$ *regular point* if df_p surjective

- $p \in M$ *critical point* if df_p not surjective
- $q \in N$ *regular value* if *every* $p \in f^{-1}(q)$ is a regualar point
- $q \in N$ *critical value* if *some* $p \in f^{-1}(q)$ is a critical point.

Note that the set of *regular points* is open and the set of *critical points* is closed.

Example (Very standard!)

$$\begin{aligned} f : \mathbb{R}^2 &\to \mathbb{R} \\ f(x,y) &:= x^2 - y^2 \end{aligned}$$

Then

$$\begin{aligned} df &= 2xdx - 2ydy, \qquad \text{or more precisely} \\ df_{(x,y)} &= 2xdx_{(x,y)} - 2ydy_{(x,y)} \end{aligned}$$

Thus (x,y) critical $\Leftrightarrow df_{(x,y)} = 0 \Leftrightarrow (x,y) = (0,0)$
All $f^{-1}(q)$ are smooth exept $f^{-1}(0)$.

Corollary 4.16 *$f : M^n \to N^k$ smooth, $q \in N$ regular value, then $f^{-1}(q)$ is a smooth submanifold of M.*

5 Lie Groups: $S^3$and SO(3)

Definition A *Lie group* is a group that has the structure of a smooth manifold such that the group operations

$$\begin{array}{rcl} G \times G & \to & G \\ (a,b) & \mapsto & ab \end{array} \qquad \begin{array}{rcl} G & \to & G \\ a & \mapsto & a^{-1} \end{array}$$

are smooth.

Example

$$\begin{aligned} \mathrm{O}(n) &:= \{A \in M^{n\times n} | A^T A = \mathbb{1}\} \\ &= \{A : \mathbb{R}^n \to \mathbb{R}^n | \langle Ax, Ay\rangle = \langle x, y\rangle \ \forall x, y \in \mathbb{R}^n\} \end{aligned}$$

$$\mathrm{SO}(n) := \mathrm{O}(n) \cap \{\det A = 1\} \quad \text{(orientation preserving)}$$

Exercise Prove $\mathrm{O}(n)$ is a Lie group by showing that $\mathbb{1}$ is a regular value of the function

$$A \in M^{n\times n} \mapsto A^T A \in M^{n\times n}_{\text{symm}}$$

Example The group of isometries of any Riemannian manifold is a Lie group (not easy at this stage).

Example

$$\mathrm{Isom}(\mathbb{R}^n) = \{x \mapsto Ax + b | A \in \mathrm{O}(n), b \in \mathbb{R}^n\}$$

Exercise What is $\mathrm{Isom}(T^2_{\text{square}})$?

5.1 Quaternions

$$\begin{aligned} \mathcal{H} &:= \{a + bi + cj + dk | a, b, c, d \in \mathbb{R}\} \\ &\cong \mathbb{R}^4 \text{ as a vector space over } \mathbb{R} \end{aligned}$$

$(\mathcal{H}, +, \cdot)$ is an *algebra* over $\mathbb{R}$.
Multiplication: 1 is multiplicative unit, and we require

$$ij = -ji = k, \ jk = -kj = i, \ ki = -ik = j$$

so that

$$\begin{aligned} (a + bi + cj + dk)(e + fi + gj + hk) &= ae - bf - cg - dh \\ &\quad +(af + be + ch - dg)i \\ &\quad +(ag + ce - bh + df)j \\ &\quad +(de + ah + bg - cf)k \end{aligned}$$

Let $u = a + bi + cj + dk$ define $\bar{u} := a - bi - cj - dk$
Check: $\bar{\bar{u}} = u$, $\overline{uv} = \bar{v}\bar{u}$.
Set $|u|^2 := u\bar{u} = a^2 + b^2 + c^2 + d^2 > 0$ (usual norm on $\mathbb{R}^4$).
Observe:

- $\frac{\bar{u}}{|u|^2}$ is the inverse of $u \neq 0$ so $\boxed{(\mathcal{H} \setminus \{0\}, \cdot) \text{ is a Lie group.}}$
- $|uv|^2 = uv\overline{uv} = uv\bar{v}\bar{u} = |v|^2|u|^2$ i.e. $|uv| = |u||v|$, "$|\cdot|$ is multiplicative".
- $S^3 := \{u \,||u| = 1\}$ is closed under multiplication and inversion, so $\boxed{(S^3, \cdot) \text{ is a Lie group}}$ called the group of unit quaternions. Note that $\boxed{S^3 \cong \mathrm{SU}(2) \cong \mathrm{Sp}(1)}$

Definition A *1-parameter subgroup* of a Lie group G is a homomorphism

$$(\mathbb{R},+) \to (G,\cdot)$$

Example

$$\begin{aligned}(\mathbb{R},+) &\to \mathbb{C} \subseteq (\mathcal{H},\cdot)\\ \theta &\mapsto e^{i\theta} := \cos\theta + i\sin\theta.\end{aligned}$$

Then $e^{i(\phi+\psi)} = e^{i\phi}\cdot e^{i\psi}$, so $\theta \mapsto e^{i\theta}$ is a 1-parameter subgroup of S^3. Now set

$$\begin{aligned}e^{j\theta} &:= \cos\theta + j\sin\theta\\ e^{k\theta} &:= \cos\theta + k\sin\theta\end{aligned}$$

These are also 1-parameter subgroups.
Take $u := ai+bj+ck$, $a^2+b^2+c^2 = 1$. Verify $u^2 = -1$ so $\{a+bu|a,b \in \mathbb{R}\} \cong \mathbb{C}$ as an algebra. Then

$$e^{u\theta} := \cos\theta + u\sin\theta$$

is also a 1-parameter sub group of S^3.

Picture of S^3

$$\begin{aligned}i &\mapsto (1,0,0)\\ j &\mapsto (0,1,0)\\ 1 &\mapsto (0,0,0)\end{aligned}$$

$$S^3 \setminus \{-1\} \xrightarrow{\cong} \mathbb{R}^3$$

In stereographic projection, the 1-parameter subgroups become lines through the origin.
All 1-parameter subgroups are equivalent, i.e. $\exists v \in S^3$ such that $v(e^{u\theta})v^{-1} = e^{i\theta}$ (Proof later).

5.2 Smooth actions, left, right, adjoint actions of a Lie group on itself

Definition G Lie group, M smooth manifold. A *smooth action* of G on M is a smooth map

$$\begin{aligned}\phi : G\times M &\to M\\ (a,x) &\mapsto \phi(a,x) \equiv \phi_a(x)\end{aligned}$$

such that

$$\begin{aligned}\phi_e &= \mathrm{id}_M\\ \phi_a \circ \phi_b &= \phi_{ab}.\end{aligned}$$

Consequences

- Each ϕ_a is diffeomorphism. To see this, compute

$$\phi_a \phi_{a^{-1}} = \phi_{aa^{-1}} = \phi_e = \mathrm{id}_M$$

  so ϕ_a is invertible with $(\phi_a)^{-1} = \phi_{a^{-1}}$, so ϕ_a is a diffeomorphism.

- ϕ yields a homomorphism

$$\begin{aligned} \phi : G &\rightarrow \mathrm{Diff}(M) \\ a &\mapsto \phi_a . \end{aligned}$$

  in agreement with our previous defintion of an action of a group on a manifold.

Definition

$$L_a : \begin{array}{c} G \to G \\ b \mapsto ab \end{array} \quad \text{left translation}$$

$$R_a : \begin{array}{c} G \to G \\ b \mapsto ba \end{array} \quad \text{right translation}$$

$a \mapsto L_a$ and $a \mapsto R_{a^{-1}}$ are smooth actions of G on itself:

$$\begin{array}{rl} L_a L_b = L_{ab}, & L_e = \mathrm{id}_G \\ R_{a^{-1}} R_{b^{-1}} = R_{(ab)^{-1}} = R_{b^{-1}a^{-1}}, & R_e = \mathrm{id}_G \end{array}$$

Note also that $L_a R_b = R_b L_a$.

Definition The *adjoint action* is defined by

$$\mathrm{Ad}_a : \begin{array}{c} G \to G \\ b \mapsto aba^{-1} \end{array} = L_a R_{a^{-1}} b = R_{a^{-1}} L_a b$$

which is also a smooth action.

Example

$$\mathbb{R}^4 \cong \mathcal{H} = \{a + bi + cj + dk\} \supseteq S^3$$

Take $u \in S^3$, then
$L_u, R_u, \mathrm{Ad}_u : \mathcal{H} \to \mathcal{H}$ are isometries, since $|uv| = |u||v| = |v|$.
Set

$$\mathbb{R}^3 := \{xi + yj + zk \mid x, y, z \in \mathbb{R}\}$$

Note that

$$T_1 S^3 \perp \mathbb{R} \cdot 1$$

where $a \in \mathbb{R}$.

Now Ad_u preserves $\mathbb{R}\cdot 1$, so Ad_u preserves $\mathbb{R}^3$, and

$$\mathrm{Ad}_u : \mathbb{R}^3 \to \mathbb{R}^3$$

is an isometry preserves O. Thus $\mathrm{Ad}_u \in \mathrm{O}(3)$ and

$$\mathrm{Ad} : S^3 \to \mathrm{O}(3)$$

is a homomorphism, i.e. $\mathrm{Ad}_u\mathrm{Ad}_v = \mathrm{Ad}_{uv}$. Now O(3) consits of two connected components, namely the orientation-preserving orthogonal transformations (SO(3)), and the orientation-reversing ones. Clearly $\mathrm{Ad} : S^3 \to \mathrm{O}(3)$ is continuous (you may check this by finding a formula for it), and S^3 is connected. Thus $\mathrm{Ad}(S^3) \subseteq \mathrm{SO}(3)$, i.e.

$$\mathrm{Ad} : S^3 \to \mathrm{SO}(3).$$

Exercise Find a formula for $\mathrm{Ad}_u \in \mathrm{SO}(3)$ and interpret it geometrically.

Kernel of Ad:

$$\begin{aligned} u \in \ker(\mathrm{Ad}) &\Leftrightarrow uvu^{-1} = v \ \forall v \in \mathbb{R}^3 \\ &\Leftrightarrow u = a \in \mathbb{R}\cdot 1 && \text{(check)} \\ &\Rightarrow u = \pm 1 \\ \ker(\mathrm{Ad}) &= \{\pm 1\} \\ \text{so } S^3/\{\pm 1\} &\cong \mathrm{SO}(3) \text{ (as a group)} \end{aligned}$$

Exercise One easily verifies: $\mathrm{Ad} : S^3 \to \mathrm{SO}(3)$ is a 2:1 covering map that takes u and $-u$ to the same point in SO(3). So

$$\mathrm{SO}(3) \overset{\text{diff}}{\cong} S^3/\{\pm 1\} \overset{\text{diff}}{\cong} \mathbb{R}P^3$$

as smooth manifolds.

Recall the following lemmas, which might help.

Lemma 5.1 *A local diffeomorphism $M \to N$ with a compact domain M is a covering map.*

Lemma 5.2 *A covering map with connected target has a constant preimage size*

$$\#\pi^{-1}(q), q \in N$$

6 Lie brackets, flows of vector fields, Lie derivatives

6.1 Vector fields

Notation:

$$X : M \to TM, X(p) \in T_pM \ \forall p$$

Let ψ be a chart $\psi : U \subseteq M \to \mathbb{R}^n$

$$X(p) = \sum_{i=1}^{n} X^i(\psi^{-1}(x^1, \dots, x^n)) \left(\frac{\partial}{\partial x^i}\right)_p$$

Warning Standard abuse of notation:

$$= \sum_{i=1}^{n} X^i(x^1, \dots, x^n) \frac{\partial}{\partial x^i}$$

where we identify p with $(x^1, \dots, x^n)$, i.e. we drop ψ.

$$\begin{aligned} C^\infty(TM) &:= \{C^\infty \text{vector fields on } M\} \\ \Gamma(TM) &:= \{\text{all vector fields on } M\} \end{aligned}$$

Also write: $C^\infty(M, TM), C^\infty(U, TM)$, where $U \subseteq M$ is open.

$$\begin{aligned} C^\infty(M) &:= \{C^\infty \text{ functions } M \to \mathbb{R}\} \\ C^0(M) &:= \{\text{continuous functions } M \to \mathbb{R}\} \\ C^1(M) &:= \{\text{continuously differentiable functions} M \to \mathbb{R}\} \\ C^k(M) &:= \{\text{functions } M \to \mathbb{R} \text{ such that all derivatives of orders} \\ &\quad\ 0, \dots, k \text{ exist and are continuous (in coordinates)}\} \end{aligned}$$

We say X is $C^k \Leftrightarrow X^i(x^1, \dots, x^n)$ are C^k

6.1.1 Lie Brackets

We wish to define $[X, Y],\ X, Y \in C^\infty(TM)$.[3]

[3] See Spivak I, 207-217

We have the map

$$\begin{aligned} C^\infty(TM) \times C^\infty(M) &\to \Gamma(M) := \{\text{functions } M \to \mathbb{R}\} \\ (X, f) &\mapsto X \cdot f \\ (X \cdot f)(p) &:= \underbrace{X(p)}_{\in T_pM} \cdot \underbrace{f}_{\in C^\infty(M)} \in \mathbb{R} \end{aligned}$$

Proposition 6.1 $X \cdot f \in C^\infty(M)$

Proof Use a chart

$$\begin{aligned} \psi : U &\to \psi(U) \subseteq \mathbb{R}^n \\ p &\mapsto (x^1, \ldots, x^n) \end{aligned}$$

Compute

$$\begin{aligned} (X \cdot f)(p) &= X(p) \cdot f \\ &= X^i(p) \left(\frac{\partial}{\partial x^i} \right)_p \cdot f \\ &= X^i \left(\psi^{-1}(x^1, \ldots, x^n) \right) \frac{\partial (f \circ \psi^{-1})}{\partial x^i} (x^1, \ldots, x^n) \end{aligned}$$

□

Consider the 2nd order differential operator $X \cdot (Y \cdot f)$, also written as XYf.

Proposition 6.2 *Let $X, Y \in C^\infty(TM)$. Then there exists a unique vector field $Z \in C^\infty(TM)$ such that*

$$Z \cdot f = (XY - YX)f, \; f \in C^\infty(M)$$

Basic idea: the 2nd order derivatives cancel.

Proof Get an expression for $(XY - YX)\,f$ in coordinates. Suppress ψ. Write

$$X = X^i \frac{\partial}{\partial x^i}, \; Y = Y^j \frac{\partial}{\partial x^j}.$$

Compute

$$\begin{aligned} XYf &= \sum_i X^i \frac{\partial}{\partial x^i} \left(\sum_j Y^j \frac{\partial f}{\partial x^j} \right) \\ &= \sum_{i,j} X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} + X^j \left(\frac{\partial Y^i}{\partial x^j} \right) \frac{\partial f}{\partial x^i} \\ YXf &= \sum_{i,j} Y^i X^j \frac{\partial^2 f}{\partial x^i \partial x^j} + Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} \end{aligned}$$

So we get

$$(XY - XY)\,f = \sum_{i,j} \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right) \frac{\partial f}{\partial x^i}.$$

Define the smooth vector field Z in the chart U by

$$Z := \sum_i Z^i \frac{\partial}{\partial x^i},\;\; Z^i := \sum_j \left(X^j \frac{\partial Y^i}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \right)$$

Then

$$Z \cdot f = (XY - YX)f$$

This shows Z is well-defined independent of parametrization, smooth and unique.

□

Definition

$$\begin{aligned} [\cdot,\cdot] : C^\infty(TM) \times C^\infty(TM) &\to C^\infty(TM) \\ [X,Y] := XY - YX & \end{aligned}$$

(as differential operator on $C^\infty(M)$) is called a *Lie bracket*.

Proposition 6.3 *Let $X, Y, Z \in C^\infty(TM), a, b \in \mathbb{R}, f, g \in C^\infty(M)$. Then*

i. $[X,Y] = -[Y,X]$ *(anticommutative)*

ii. $[aX + bY, Z] = a[X,Z] + b[Y,Z]$ *(bilinear)*

iii. $[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0$ *(Jacobi identity)*

iv. $[fX, gY] = fg[X,Y] + f(X \cdot g)Y - g(Y \cdot f)X$

Proof Jacobi Identity

$$\begin{array}{rcccl} [[X,Y],Z] & = & [XY - YX, Z] & = & (XY - YX)Z - Z(XY - YX) \\ [[Y,Z],X] & = & [YZ - ZY, X] & = & (YZ - ZY)X - X(YZ - ZY) \\ [[Z,X],Y] & = & [ZX - XZ, Y] & = & (ZX - XZ)Y - Y(ZX - XZ) \\ & & & & \text{sum} = 0 \end{array}$$

□

Definition A vector space V equipped with a bracket $[\cdot,\cdot] : V \times V \to V$ satisfying i, ii, iii is called a *Lie algebra*.
So $C^\infty(TM)$ forms a Lie algebra.

Example Another famous Lie algebra:
V vector space over a field $\mathbb{K}$

$$\begin{aligned} \mathrm{End}_{\mathbb{K}}(V) &:= \mathrm{Hom}_{\mathbb{K}}(V,V) \\ [A,B] &:= AB - BA \end{aligned}$$

$(\mathrm{End}_{\mathbb{K}}(V), [\cdot,\cdot])$ is a Lie algebra.

Example $M^{n\times n}(\mathbb{R}), M^{n\times n}(\mathbb{C})$.

Relationships between the two kinds of $[\cdot,\cdot]$ occurs via the *Lie Algebra of (matrix) Lie groups.*

6.2 Integral curves and flows of vector fields[4]

Definition An *integral curve of* X is a path $\gamma : [a,b] \to M$ such that

$$\dot\gamma(t) = X(\gamma(t)),\ t \in [a,b].$$

In coordinates, this is an $n\times n$ first order ODE system. We write and obtain:

$$\begin{aligned} \gamma(t) &= \left(x^1(t),\ldots,x^n(t)\right) \in U \subseteq \mathbb{R}^n \\ \frac{dx^1}{dt} &= X^1\left(x^1(t),\ldots,x^n(t)\right) \\ &\vdots \\ \frac{dx^n}{dt} &= X^n\left(x^1(t),\ldots,x^n(t)\right),\ a \le t \le b. \end{aligned}$$

6.2.1 Existence, Uniquenes and smooth dependence on initial data

.
Consider the ODE system

$$(*)\begin{cases} \frac{d\gamma(t)}{dt} = X(\gamma(t)) & -a < t < b, a, b > 0 \\ \gamma(0) = p & \text{require: } \gamma \text{ is } C^1 \end{cases}$$

Theorem 6.4 (Short-term existence, uniqueness, regularity for γ) *Let $X \in C^\infty(TM)$. Then*

*i. $\exists\delta > 0$ such that $(*)$ has a C^1 solution defined for $-\delta < t < \delta$. (*Existence*)*

[4]See Spivak I Chap. 5.

*ii. Any C^1 solution of $(*)$ is C^∞ (*Regularity*)*

*iii. Any two C^1 solutions of $(*)$ on $(-a,b),(-c,d), a,b,c,d > 0$ agree on their commmon interval of definition $(-a,b)\cap(-c,d)$. (*Uniqueness*)*

Proof

Analysis: Either Inverse Function Theorem on Banach spaces, or a successive approximation method[5].

ii. *Exercise*

$\square$

Remark $X \in C^k \Rightarrow$ Theorem holds but with γ in C^{k+1}

Dependence on Initial Conditions

Write $\gamma_x(t) \equiv \phi(x,t) \equiv \phi^t(x)$ (integral curve with initial point $\gamma_x(0) = x$). The equation $(*)$ becomes

$$(*)' \begin{cases} \frac{\partial \phi(x,t)}{\partial t} & = & X(\phi(x,t)), & x \in U, -a < t < b \\ \phi(x,0) & = & x, & x \in U. \end{cases}$$

Theorem 6.5 (Dependence on initial conditions of ϕ) *Let $X \in C^\infty(TM), p \in M$.*

i. $\exists U \ni p, \delta > 0$ and a function (C^1 in t) $\phi : U \times (-\delta,\delta) \to M$ that solves $(*)'$.

ii. Any solution of $(*)'$ that is C^1 in t is C^∞ in x and t.

iii. Any two solutions $\phi : U \times (-a,b) \to M, \psi : V \times (-c,d) \to M$ agree on the intersection of their domains.

Remark $X \in C^k \Rightarrow \phi$ is C^k in (x,t) (recall from above that ϕ is C^{k+1} in t).

New point of view:

$$\phi_t : \underbrace{U}_{\subseteq M} \to \underbrace{\phi_t(U)}_{\subseteq M}$$

The family $(\phi_t)_{-a<t<b}$ is called a *local flow of* X.

Notation:

$A \subset\subset B$ means $\overline{A}$ is compact and $\overline{A} \subseteq B$, read "*A compactly contained in B*". If $\overline{A}$ is compact, we say A is *precompact*.

[5]See Lang reference in Spivak I chap 5. Alternately see Rivieère's differential geometry problem last year.

Theorem 6.6 (Larger U, smaller δ) *For any $U \subset\subset M$ $\exists \delta > 0$ such that the local flow is defined on $U \times (-\delta, \delta)$.*

Proof By compactness of $\bar{U}$, we may cover $\bar{U}$ by finitely many open sets $V_1, \ldots, V_n$ such that there are flows (solving $(*)'$)

$$\phi_i : V_i \times (-\delta_i, \delta_i) \to M.$$

Set $\delta := \min \delta_i > 0$. Define

$$\phi : U \times (-\delta, \delta) \to M$$

by:

$$\phi := \phi_i \text{ on } V_i \times (-\delta, \delta)$$

(Consistent by uniqueness assertion (iii) in previous Theorem)

□

Theorem 6.7 (Pseudogroup Property) *If $\phi^t \circ \phi^s$ is defined on U for $|s| < S, |t| < T$, then ϕ^u is defined on U for $|u| < S + T$ and*

$$\phi^{t+s} = \phi^t \circ \phi^s \text{ on } U$$

If $\phi_t : M \to M$ exists for all time $t \in \mathbb{R}$, then ϕ_t is called a *complete flow*. Note that ϕ_t injective $\Leftrightarrow$ uniqueness of initial value problem for *backwards* flow.

Proof Fix $|s| < S, |t| < T$. Combine the two paths via

$$\alpha(u) := \begin{cases} \gamma_x(u) & 0 \leq u \leq s \\ \gamma_{\gamma_x(s)}(u - s) & s \leq u \leq s + t \end{cases}$$

Note that

$$\gamma_x(s) = y = \gamma_{\gamma_x}(0) \quad \Rightarrow \quad \alpha \text{ is } C^0$$
$$\dot{\gamma}_x(s) \overset{(*)}{=} X(y) \overset{(*)}{=} \dot{\gamma}_{\gamma_x(s)} \quad \Rightarrow \quad \alpha \text{ is } C^1$$

Also α solves $(*)$. So define (extend) γ via $\gamma_x(u) := \alpha(u), 0 \leq u \leq t + s$.

Remark (Used in above step) If $\gamma(u), a \leq u \leq b$ solves ODE $(*)$, then so does the time shifted curve $\gamma(u - k), a + k \leq u \leq b + k$.

So $\phi^u : U \to M$ exists, $0 \le u \le t+s$ and $\phi^t \circ \phi^s = \phi^{t+s}$. Specifically:

$$\begin{aligned}\phi^t \circ \phi^s(x) &= \phi^t(\phi^s(x)) \\ &= \phi^t(\gamma_x(s)) \\ &= \gamma_{\gamma_x(s)}(t) \\ &= \alpha(s+t) \\ &= \gamma_x(s+t) \\ &= \phi^{s+t}(x).\end{aligned}$$

□

Corollary 6.8 *Assume U open and ϕ_t exists on U. Then: $\phi_t(U)$ is open and $\phi_t|U : U \to \phi_t(U)$ is a diffeomorphism.*

Proof

i. Assume first that ϕ_t is complete. Then by previous Theorem:

$$\phi_{-t} \circ \phi_t = \phi_{-t+t} = \phi_0 = \mathrm{id}_M.$$

So ϕ_t is invertible with inverse

$$(\phi_t)^{-1} = \phi_{-t} : M \to M$$

and ϕ_{-t} is smooth, so $\Rightarrow \phi_t : M \to M$ is a diffeomorphism and $\phi_t(U)$ open for any open $U \subseteq M$ and $\phi_t|U : U \to \phi_t(U)$ is a diffeomorphism.

ii. Next we do the global case (when ϕ_t is not complete).

Let $U \subset\subset M$ and try for small t. Choose V open such that $U \subset\subset V \subset\subset M$. Choose δ so small that

$$\begin{aligned}\phi :&\quad U \times [0,\delta] &\to V \\ \phi :&\quad V \times [-\delta, 0] &\to M\end{aligned}$$

are defined. Then

$$\phi_{-\delta} \circ \phi_\delta : U \to M$$

is defined, so by above Theorem $\phi_{-\delta} \circ \phi_\delta = \mathrm{id}$ on U. It follows that $\phi_\delta|U$ is a local diffeomorphism, $\phi_\delta(U)$ is open, and $\phi_\delta|U$ is a diffeomorphism.

Lemma 6.9 *A smooth map*

$$\phi : U \to M \ (U \text{ open})$$

with a smooth left *inverse* $\psi : A \supseteq \phi(U) \to M, A$ *open*

$$\psi \circ \phi = id_U$$

is a diffeomorphism and $\phi(U)$ is open.

iii. Next, let $U \subset\subset M$ and let $t > 0$ be an arbitrary time such that ϕ_t exsists on $\bar{U}$. Choose V open such that

$$\phi(\bar{U} \times [0, t]) \subset\subset V \subset\subset M.$$

For δ small enough, ϕ_δ will be defined on V and $\phi_\delta : V \to \phi_\delta(V)$ will be a diffeomorphism. Making δ slightly smaller, we can arrange

$$t = k\delta, \phi_t = \underbrace{\phi_\delta \circ \cdots \circ \phi_\delta}_{k}$$

on U. Thus $\phi_t|U$ is a diffeomorphism onto the open set $\phi_t(U)$.

iv. Now let $U \subseteq M$ be an arbitrary open set and let ϕ_t be defined on U. For ever $V \subset\subset U$, $\phi_t(V)$ is open and $\phi_t|V : V \to \phi_t(V)$ is a diffeomorphism. It follows that $\phi_t(U)$ is open and $\phi_t|U : U \to \phi_t(U)$ is a diffeomorphism.

Get in succession:

$$\phi_\delta : V \to \phi_\delta(V) \text{ diffeomorphism, } \phi_\delta(V) \text{ open}$$

$$U \subseteq V, \text{ so } \phi_\delta(U) \text{ is open}$$
$$\phi_\delta|U : U \to \phi_\delta(U) \text{ diffeomorphism}$$
$$\phi_\delta(U) \subseteq V, \text{ so} \phi_\delta(\phi_\delta(U)) \text{ is open}$$
$$\phi_\delta|\phi_\delta(U) : \phi_\delta(U) \to \phi(\phi_\delta(U)) \text{ diffeomorphism}$$
$$\text{Thus } \phi_{2\delta} = \phi_\delta \circ \phi_\delta : U \to \phi_\delta \circ \phi_\delta(U) \text{ diffeomorphism}$$
$$\text{Induction } \Rightarrow \phi_t : U \to \phi_t(U) \text{ diffeomorphic}$$
$$\phi_t(U) \text{ is open.}$$

□

Remark on uniqueness

$$\dot{x}(t) = X(x(t)), x(t) \in U \subseteq \mathbb{R}^n$$

Sufficient conditions for uniqueness: X is *Lipschitz*.

Example Fix $0 < \alpha < 1$. Consider

$$\begin{cases} \dot{x} &= x(t)^\alpha, \quad t \geq 0 \\ x(0) &= 0. \end{cases}$$

Solving, we find a solution

$$x(t) = ((1-\alpha)t)^{\frac{1}{1-\alpha}}, t \geq 0$$

In fact, we have *two* solutions

$$\begin{aligned} x(t) &:= \begin{cases} 0 & t \leq 0 \\ ((1-\alpha)t)^{\frac{1}{1-\alpha}}, & t \geq 0 \end{cases} \\ y(t) &:= 0 \quad t \in \mathbb{R}. \end{aligned}$$

Since $\frac{1}{1-\alpha} > 1$, $x(t)$ is C^1 in t.

Question How far can we extend the flow?

Definition A vector field is called *complete* if it possesses a flow $\phi_t : M \to M$ defined for all $-\infty < t < \infty$.

Remark Then $t \mapsto \phi_t$ defines a 1-parameter subgroup of $\mathrm{Diff}(M)$, or equivalently, a smooth action of $\mathbb{R}$ on M.

Example

$$X(x, y) := (x, -y) \text{ on } \mathbb{R}^2$$

A typical solution traces out a curve: $xy = \text{const}$, and has the form

$$\gamma(t) := \left(C_1 e^t, C_2 e^{-t}\right), t \in \mathbb{R}.$$

So this X is complete.

Example

$$\dot{x} = x^2, x(t) \in M := \mathbb{R}, X(x) = x^2 \frac{\partial}{\partial x}.$$

Solution: $x(t) = \frac{1}{C-t}, -\infty < t < c$ (or $c < t < \infty$) *So this X is incomplete.*

Example Clearly

$$\dot{y} = 1, y(t) \in N := (-\infty, 0)$$

is incomplete

Transform the equation to $x = -\frac{1}{y}, \dot{x} = \frac{\dot{y}}{y^2} = \frac{1}{(1/x)^2} = x^2$. It becomes equivalent to the previous problem, with $M = (0, \infty)$. In both cases, the trajectory runs off the end of the manifold in finite time

Example

$$X = \frac{\partial}{\partial x}, U \subseteq \mathbb{R}^2$$

Typically incomplete.

Corollary 6.10 (to group property and short-time existence) *If $\phi : U \times [0, T) \to M$ and $\phi(U \times [0, T)) \subset\subset M$ then ϕ can be extended to a solution $\phi : U \times [0, T + \delta) \to M$ for some $\delta > 0$.*

Proof Pick V such that

$$\phi(U \times [0, T)) \subseteq V \subset\subset M$$

ϕ_t is defined on V for $0 \leq t < T$ and $\delta > 0$ such that there is a local flow

$$\phi : V \times [0, \delta) \to M.$$

Then ϕ_s is defined on V for $0 \leq s < \delta$. Apply the group property to yield

$$\phi^{s+t} = \phi^s \circ \phi^t = \phi^u, \ 0 \leq u < T + \delta,$$

i.e. we can extend ϕ to

$$\phi : U \times [0, T + \delta) \to M.$$

□

Significance A trajectory $\gamma(t)$ can be continued as long as it stays in a compact set of M. (i.e. if $[0, T)$ is the *maximum* time of existence of $\gamma(t)$, then $\gamma(t)$ must leave every compact set of M.)

Corollary 6.11 *If M is compact, then every smooth vector field on M is complete.*

Theorem 6.12 *If $X \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ has* at most linear growth, *i.e.*

$$|X(x)| \leq C_1 |x| + C_2, x \in \mathbb{R}^n,$$

then X is complete.

Example

$$\dot{x} = x, \ \dot{x} = x + 1, \ \dot{x} = \begin{cases} \log x, & x \geq 1 \\ \dots & x \leq 1. \end{cases}$$

Proof Let $\dot{x}(t) = X(x(t)), x(t) \in \mathbb{R}^n$, $X : \mathbb{R}^n \to \mathbb{R}^n$.
It follows:

$$\begin{aligned} \frac{d}{dt}|x(t)| &= \langle \frac{dx}{dt}, \frac{x}{|x|} \rangle \\ &\leq \left| \frac{dx}{dt} \right| \\ &= |X(x(t))| \\ &\leq C_1 |x(t)| + C_2 \end{aligned}$$

Compare $|x(t)|$ to the solution of

$$\begin{cases} \frac{da}{dt} &= C_1 a + C_2, \quad a(t) \in \mathbb{R} \\ a(0) &= |x(0)| \end{cases}$$

□

Lemma 6.13

$$|x(t)| \leq a(t), t \geq 0$$

Proof Let $b(t) := |x(t)| - a(t)$. Compute

$$\begin{aligned} \frac{db}{dt} &= \frac{d|x(t)|}{dt} - \frac{da}{dt} \\ &\leq C_1|x| + C_2 - (C_1 a + C_2) \\ &= C_1 b. \end{aligned}$$

So $b(t)$ solves:

$$\begin{cases} b(0) &= 0 \\ \frac{db(t)}{dt} &\leq C_1 b(t) \end{cases}$$

Claim

$$b(t) \leq 0 \ \forall t \geq 0.$$

To see this, we argue as follows.
On the open set $I \subseteq \mathbb{R}$ where we compute that $b(t) > 0$, set $B(t) := \log b(t)$.
Write $I = \cup_\alpha (a_\alpha, b_\alpha)$, where $(a_\alpha, b_\alpha) \cap (a_\beta, b_\beta) = \varnothing$. $\frac{dB}{dt} \leq C_1$.

Now $B(t) \to -\infty$ as $t \to {a_\alpha}^t$ inside (a_α, b_α)
so $B(t) - C_1 t \to -\infty$ as $t \to {a_\alpha}^t$
but $B(t) - C_1 t$ is nonincreasing. *This is impossible.* Thus $I = \varnothing$.

This proves the claim.

Upshot:

$$|x(t)| \le a(t) = \left(|x(0)| + \frac{C_2}{C_1}\right) e^{C_1 t} - \frac{C_2}{C_1}$$

which is finite, as long as $0 \le t < T$. This shows: $x\left([0,T)\right)$ lies in a compact subset of $\mathbb{R}^n$ for any $T < \infty$. Thus: $x(t)$ can be continued forever (i.e. $\forall t$).

□

Theorem 6.14 *Let $X \in C^\infty(TM)$. Fix $p \in M$. If $X(p) \neq 0$, then there are coordinates $(x^1, \ldots, x^n)$ near p with $X(q) = \left(\frac{\partial}{\partial x^1}\right)_q$ for all q near p.*

Meaning: There are no local invariants of nonzero vector fields (they are all the same, locally).

Proof Choose coords $y^1, \ldots, y^n$ on a small neighborhood $U \ni p$ such that

$$X(p) = \left(\frac{\partial}{\partial y^1}\right)_p, \; p = (0, \ldots, 0).$$

We have

$$\begin{array}{rccc} \phi: & U \times (-\varepsilon, \varepsilon) & \to & M \\ & (y^1, \ldots, y^n, t) & \mapsto & (\phi^1, \ldots, \phi^n). \end{array}$$

Now $N := U \cap \{y^1 = 0\}$ is a submanifold of M passing through p. Define

$$\begin{array}{rccc} \psi := \phi|_{N \times (-\varepsilon, \varepsilon)}: & N \times (-\varepsilon, \varepsilon) & \to & M \\ & (y^2, \ldots, y^n, t) & \mapsto & (\psi^1, \ldots, \psi^n) \end{array}$$

Concretely. $\psi^i(y^2, \ldots, y^n, t) := \phi^i(0, y^2, \ldots, y^n, t)$. We wish to apply the Inverse Function Theorem to ψ at the point

$$(p, 0) \in N \times (-\varepsilon, \varepsilon), \; \psi(p, 0) = p,$$

to prove that $(y^2, \ldots, y^n, t)$ can be taken as coordinates on M near p. For $(q, t) \in N \times (-\varepsilon, \varepsilon)$:

$$(d\psi)_{(q,t)} : T_{(q,t)}\left(N \times (-\varepsilon, \varepsilon)\right) = T_q N \times \mathbb{R} \to T_{\psi(q,t)} M$$

Compute for $(q, t) \in N \times (-\varepsilon, \varepsilon)$::

$$\begin{aligned} (d\psi)_{(q,t)}\left(\left(\frac{\partial}{\partial t}\right)_{(q,t)}\right) &= \frac{\partial \psi}{\partial t}(q, t) \\ &= \frac{\partial \phi}{\partial t}(q, t) \\ &= X\left(\phi(q, t)\right) \\ &= X\left(\psi(q, t)\right). \end{aligned}$$

At $(p, 0)$, we have:

$$\psi(p,0) = p$$

$$d\psi_{(p,0)} : \begin{array}{ccc} T_qN \times \mathbb{R} & \to & T_pM \\ \frac{\partial}{\partial y^2}, \dots, \frac{\partial}{\partial y^n}, \frac{\partial}{\partial t} & & \frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^n} \end{array}.$$

We get

$$\left(\frac{\partial}{\partial t}\right)_{p,0} \mapsto X(p) = \left(\frac{\partial}{\partial y^1}\right)_p \qquad \text{(by above)}$$

and

$$\left(\frac{\partial}{\partial y^i}\right)_{(p,0)} \mapsto \left(\frac{\partial}{\partial y^i}\right)_p \quad i = 2, \dots, n$$

since $\psi|N \times \{0\}$ is just the inclusion $N \to M$. Thus $(d\psi)_{(p,0)}$ is an isomorphism, so by Inverse Function Theorem,

$$\psi : V \times (-\delta, \delta) \to W \subseteq M$$

is a diffeomorphism for some small $p \in V \subseteq N, p \in W \subseteq M, \delta > 0$. So we may take $(y^2, \dots, y^n, t)$ as coordinates on W. For $r := \psi(q,t) \in W$, we get:

$$\begin{aligned} \left(\frac{\partial}{\partial t}\right)_r &= (d\psi)_{(q,t)}\left(\left(\frac{\partial}{\partial t}\right)_{q,t}\right) \\ &= X(\psi(q,t)) \\ &= X(r) \end{aligned}$$

□

Definition (Codimension) Let M^n be a manifold, $N^k \subseteq M^n$ a submanifold of M. Then the codimension of N inside M is $\dim M - \dim N = n - k$.

6.3 Lie Derivatives

Pushforward and Pullback of Vector fields

$$f : M \to N$$

Definition (Pushforward) Given $X \in C^\infty(TM)$ we wish to produce $f_*(X) \in C^\infty(TN)$

If f is *bijective*, define the *pushforward of X via f* by

$$f_*(X)(q) := df_{f^{-1}(q)}\left(X(f^{-1}(q))\right) \in T_q(N) \;\forall q \in N.$$

Definition (Pullback)

$$f^*(X) \in C^\infty(TM) \leftarrow X \in C^\infty(TN)$$

If $df_p : T_pM \to T_{f(p)}N$ is bijective $\forall p \in M$, define the *pullback of X via f* by

$$f^*(X)(p) := (df_p)^{-1}(X(f(p)))$$

Easy case: f is a diffeomorphism $\Rightarrow f_*, f^*$ are both defined.

Proposition 6.15 *(Exercise)*

i. $f_*(X), f^*(Y)$ *are smooth if* X, Y *are smooth*

ii. Given

$$M \xrightarrow{f} N \xrightarrow{g} P, \quad (g \circ f : M \to P)$$

$$X \in C^\infty(TM), Z \in C^\infty(TP)$$

We have

$$\begin{aligned} g_* f_* X &= (g \circ f)_*(X) \\ f^* g^* Z &= (g \circ f)^*(Z) \end{aligned}$$

iii. f *a diffeomorphism* $\Rightarrow f^*Y = (f^{-1})_*Y, f_*X = (f^{-1})^*X \;\; f^*f_*X = X, f_*f^*Y = Y.$

Lie Derivative

We wish to define $L_XY,\ X, Y \in C^\infty(TM)$. We wish to differentiate Y in the direction of X.
Let $X, Y \in C^\infty(TM)$. Let ϕ_t be the flow of X. Idea: look forward along the flow of X to see how Y is changing. We must pull back Y by ϕ_t to make the comparison.
$\phi_t^*(Y)$: family of vector fields on M, with starting value

$$\phi_0^*(Y) = \mathrm{id}_M^*(Y) = Y \ (t = 0).$$

Definition

$$\begin{aligned} L_XY(p) &:= \left.\frac{d}{dt}\right|_0 \phi_t^*(Y)(p) = \lim_{t \to 0} \frac{\phi_t^*(Y)(p) - Y(p)}{t} \\ &= \lim_{t \to 0} \frac{(d\phi_p^t)^{-1}(Y(\phi_t(p))) - Y(p)}{t} \in T_pM \end{aligned}$$

The subtraction is permitted because $\phi_t^*(Y)(p)$ and $Y(p)$ both live in T_pM.

Proposition 6.16 *If $X, Y \in C^\infty(TM)$, then the defintion exists, $L_X Y$ is a smooth vector field, and*

$$L_X Y = [X, Y]. \tag{†}$$

Proposition 6.17

i. $f^*(L_X Y) = L_{f^*X} f^*Y$

ii. $f^*[X, Y] = [f^*X, f^*Y]$ *if df_p is bijective $\forall p$, i.e f is a local diffeomorphism.*

We leave ii as an exercise.

Proof of i)
Assume f is any local diffeomorphism, work in a small neighborhood and f becomes a diffeomorphism.

$$\begin{array}{cccccc} N & X & Y & L_X Y & \phi_t & \text{flow of } X \\ f\big\uparrow & \big\downarrow f^* & \big\downarrow f^* & \big\downarrow f^* & \big\downarrow f^* & \\ M & \tilde{X} & \tilde{Y} & \widetilde{L_X Y} & \tilde{\phi}_t & \text{flow of } Y \text{ (proof below)} \end{array}$$

To prove: $\widetilde{L_X Y} = L_{\tilde{X}} \tilde{Y}$.

Claim The pullback of a flow of X is a flow of the pullback of X

Proof (of claim)
For simplicity, just do the case where X is complete.

$$\begin{array}{ccc} N & \xrightarrow{\phi_t} & N \\ f\big\uparrow & & \big\uparrow f \\ M & \xrightarrow{\tilde{\phi}_t} & M \end{array}$$

Let ϕ_t be the flow of X. Then

$$\tilde{\phi}_t := f^{-1} \circ \phi_t \circ f := f^*(\phi_t)$$

is the flow of $f^*(X)$
Note $d(f^{-1})_q = \left((df)_{f^{-1}(q)}\right)^{-1}$, where $q = f(p)$.

Compute

$$
\begin{aligned}
\frac{\partial}{\partial t}\tilde{\phi}_t(p) &= \frac{\partial}{\partial t} f^{-1}\circ\phi_t\circ f(p) \\
&= d(f^{-1})_{\phi_t(f(p))}\left(\frac{\partial}{\partial t}\left(\phi_t(f(p))\right)\right) \\
&= \left(df_{f^{-1}(\phi_t(f(p)))}\right)^{-1}\left(X(\phi_t(f(p)))\right) \\
&= \left(df_{\tilde{\phi}_t(p)}\right)^{-1}\left(X(f(\underbrace{f^{-1}(\phi_t(f(p)))}_{\tilde{\phi}_t(p)})\right) \\
&= f^*(X)\left(\tilde{\phi}_t(p)\right) \\
&= \tilde{X}\left(\tilde{\phi}_t(p)\right).
\end{aligned}
$$

□

We return to the proof of $L_{\tilde{X}}\tilde{Y} = \widetilde{L_X Y}$. Compute

$$
\begin{aligned}
L_{\tilde{X}}\tilde{Y} &= \left.\frac{\partial}{\partial t}\right|_0 \tilde{\phi}_t^*(\tilde{Y}) \\
&= \left.\frac{\partial}{\partial t}\right|_0 (f^{-1}\circ\phi_t\circ f)^*(f^*Y) \\
&= \left.\frac{\partial}{\partial t}\right|_0 f^*\phi_t^*(f^{-1})^*f^*Y \\
&= f^*\left.\frac{\partial}{\partial t}\right|_0 (\phi_t^*Y) \\
&= f^*(L_X Y) \\
&=: \widetilde{L_X Y}
\end{aligned}
$$

□

Proof of †. Both sides are well-defined, coordinate free concepts, as shown by the Lemma. Thus it suffices to prove claim (†) in a chart, $U \subseteq \mathbb{R}^n$. That is, we prove it for the push forwards of X and Y on $V \subseteq M$ to $U \subseteq \mathbb{R}^n$ via the chart $\psi : V \to U$, then pull back the result to M.
So let $X, Y \in C^\infty(U, \mathbb{R}^n), U \subseteq \mathbb{R}^n$ open, fix $p \in U$. Let ϕ_t be a local flow of X near p. (defined on $p \in V \subset\subset U, -\delta < t < \delta$).

Compute:

$$\begin{aligned} Z(p) &:= L_X Y(p) = \left.\frac{d}{dt}\right|_0 \phi_t^*(Y)(p) \\ &= \left.\frac{d}{dt}\right|_0 (d\phi_t(p))^{-1} (Y(\phi_t(p))) \end{aligned}$$

Where

$$d\phi_t(p) : T_p U = \mathbb{R}^n \to T_{\phi_t(p)} U = \mathbb{R}^n.$$

□

Lemma 6.18 *Let $A(t) : V \to W$ be a smooth family of invertible linear maps. Then*

$$\frac{d}{dt} A(t)^{-1} = -A(t)^{-1} \frac{d}{dt} A(t) \circ A(t)^{-1}$$

Proof Write $B(t) := A(t)^{-1}$ so differentiate $A(t) \circ B(t) = I$ get $A'(t) \circ B(t) + A(t) \circ B'(t) = 0$. Now solve for $B'(t)$:

$$B'(t) = -A(t)^{-1} \circ A'(t) \circ A(t)^{-1}.$$

□

Continue with the computation of $L_X Y$, we get:

$$\begin{aligned} Z(p) &= \left.\frac{d}{dt}\right|_0 (d\phi_t(p))^{-1} (Y(\phi_0(p))) + \left.\frac{d}{dt}\right|_0 (d\phi_0(p))^{-1} (Y(\phi_t(p))) \\ &= -d\phi_0(p)^{-1} \left.\frac{d}{dt}\right|_0 d\phi_t(p) d\phi_0(p)^{-1}(Y(p)) + \left.\frac{d}{dt}\right|_0 Y(\phi_t(p)) \\ &= -\left.\frac{d}{dt}\right|_0 d\phi_t(p)(Y(p)) + \left.\frac{d}{dt}\right|_0 Y(\phi_t(p)) \end{aligned}$$

We used the fact that $\left.\frac{d}{dt}\right|_0 f(t,0) = \left.\frac{d}{dt}\right|_0 f(t,t) - \left.\frac{d}{dt}\right|_0 f(0,t)$.
Now we use the coordinates of $\mathbb{R}^n$ explicitly[6]. Write

$$Z = (Z^i) \in \mathbb{R}^n$$

$$d\phi_t(p) = \left(\frac{\partial \phi_t^i(p)}{\partial x^j}\right) : \mathbb{R}^n \to \mathbb{R}^n$$

[6]They were already used subtly in the first line above, by subtracting $Y(p)$ from $Y(\phi_t(p))$

$$X = (X^i),\ X^i(p) = \left.\frac{\partial \phi_t^i(p)}{\partial t}\right|_0, Y = (Y^i).$$

Compute

$$\begin{aligned} Z^i &= -\left.\frac{\partial}{\partial t}\right|_0 \frac{\partial \phi_t^i(p)}{\partial x^j} Y^j(p) + \frac{\partial Y^i}{\partial x^j}(p) \left.\frac{\partial \phi_t^j}{\partial t}\right|_0 (p) \\ &= -\frac{\partial}{\partial x^j} \left.\frac{\partial \phi_t^i(p)}{\partial t}\right|_0 Y^j(p) + \frac{\partial Y^i}{\partial x^j}(p) X^j(p) \\ &= -\frac{\partial X^i}{\partial x^j} Y^j(p) + \frac{\partial Y^i}{\partial x^j} X^j(p) = [X, Y]^i \end{aligned}$$

So we get the important formula:

$$\boxed{(L_X Y)^i = -\tfrac{\partial X^i}{\partial x^j} Y^j + \tfrac{\partial Y^i}{\partial x^j} X^j = [X, Y]^i}$$

i.e. $L_X Y = [X, Y]$, as desired.

□

Corollary 6.19

$$L_X Y = -L_Y X.$$

Interpretation of $[X, Y]$ via the flows of X and Y

Construction: Fix p. Set

$$f(s, t) := \psi_{-s} \circ \phi_{-t} \circ \psi_s \circ \phi_t(p)$$

Where ϕ_t is the flow of X and ψ_s the flow of Y.
Question: How does $f(s, t)$ differ from p?

Theorem 6.20 *In any coordinate system*

$$f(s, t) = p + st[X, Y](p) + O\left((|s| + |t|)^3\right)$$

(for s, t small).

This says: the flows commute up to 1st oder, and the (2nd order) discrepancy is measured by $[X, Y]$.

Proof Exercise.

□

Theorem 6.21
$$[X,Y]=0 \Leftrightarrow \psi_s\circ\phi_t=\phi_t\circ\psi_s$$

Proof $\Leftarrow$ by above (differentiation)

$\Rightarrow$ exercise (integration)

□

Definition If $[X,Y]=0$, we say X,Y *commute.*

Example

- $[\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}]=0$
- $[\frac{\partial}{\partial x},x\frac{\partial}{\partial x}+\frac{\partial}{\partial y}]=[\frac{\partial}{\partial x},x\frac{\partial}{\partial x}]+[\frac{\partial}{\partial x},\frac{\partial}{\partial y}]=\frac{\partial x}{\partial x}\frac{\partial}{\partial x}-x\frac{\partial}{\partial x}\frac{\partial}{\partial x}=\frac{\partial}{\partial x}$

Corollary 6.22 *Fix p. If $X(p),Y(p)$ are linearly independent and $[X,Y]=0$ near p, then there are coordinates near p with*
$$X=\frac{\partial}{\partial x^1},\ Y=\frac{\partial}{\partial x^2}$$

Proof of Corollary Take s,t as coordinates, defining
$$\Psi(s,t):=\psi_s(\phi_t(p))\ \ (=\phi_t(\psi_s(p)))$$
$$\Psi:\mathbb{R}^2\supseteq U\ni(0,0)\to M\quad\text{smooth}$$
We compute
$$\begin{aligned} d\Psi_{(s,t)}\left(\frac{\partial}{\partial s}\right) &= \frac{\partial}{\partial s}\Psi(s,t)\\ &= \frac{\partial}{\partial s}\psi_s\left(\phi_t(p)\right)\\ &= Y\left(\psi_s\left(\phi_t(p)\right)\right)\\ &= Y\left(\Psi(s,t)\right)\end{aligned}$$
Similarly here we use, that the flows commute
$$d\Psi_{(s,t)}\left(\frac{\partial}{\partial t}\right)=X\left(\Psi(s,t)\right).$$

Note

$$d\Psi_{(0,0)}: \begin{array}{ccc} \frac{\partial}{\partial s} & \mapsto & Y(p) \\ \frac{\partial}{\partial t} & \mapsto & X(p) \end{array} \Bigg\} \text{linearly independant}$$

so $d\Psi_{(0,0)}$ is an isomorphism, so Ψ is a diffeomorphism near $(0,0)$, so s,t are valid smooth coordinates on a neighborhood of p, and the coordinate vector field $\left(\frac{\partial}{\partial s}\right)_q$ (for $q = \Psi(s,t)$ near p) is given by $d\Psi_{(s,t)}\left(\frac{\partial}{\partial s}\right)$, which is $Y(q)$ as we have just seen. Similarly, $\left(\frac{\partial}{\partial t}\right)_q = X(q)$.

□

Interpretations of Jacobi Identity

Recall the Jacobi identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

i. Rewrite the Jacobi identity as

$$L_X[Y,Z] = [Y, L_X Z] + [L_X Y, Z]$$

A Leibniz rule relating L_X to the $[\cdot,\cdot]$ product. One says: L_X is a *derivation* for $[\cdot,\cdot]$.

ii. Rewrite the Jacobi identity as

$$L_{[X,Y]}Z = L_X L_Y Z - L_Y L_X Z$$

i.e.

$$L_{[X,Y]} = L_X \circ L_Y - L_Y \circ L_X \ (=: [[L_X, L_Y]]).$$

The later bracket operator, $[[\cdot,\cdot]]$ is the anticommutator defined on any algebra of endomorphisms. So

$$L: \begin{array}{ll} C^\infty(TM) & \to \text{End}(C^\infty(TM)) \\ X & \mapsto L_X \end{array}$$

so L is a *bracket homomorphism* from $(C^\infty(TM), [\cdot,\cdot])$ to $(\text{End}(C^\infty(TM)), [[\cdot,\cdot]])$

7 Riemannian Metrics

Do Carmo Chap 1

Definition Let M be a smooth manifold. A *(smooth) Riemannian metric* on M is a choice of inner product

$$\langle \cdot, \cdot \rangle_p : T_pM \times T_pM \to \mathbb{R}$$

on each tangent space, that is smooth in the sense defined below.

- bilinear, symmetric
- positive definite, i.e.

$$\langle X, X \rangle_p > 0, \ \forall X \neq 0.$$

Notation: Also write g_p or $g(p)$ for $\langle \cdot, \cdot \rangle_p$. Write g for the map $p \mapsto g_p$. We call (M, g) a *Riemannian manifold.*

Coordinate Expression

Let $U \subseteq M, X = X^i \frac{\partial}{\partial x^i}, Y = Y^j \frac{\partial}{\partial x^j}$ on U.
Write

$$\begin{aligned} g(p)\left(X(p), Y(p)\right) &= g(p)\left(X^i(p)\left(\frac{\partial}{\partial x^i}\right)_p, Y^j(p)\left(\frac{\partial}{\partial x^j}\right)_p\right) \\ &= X^i(p)Y^j(p)g(p)\left(\left(\frac{\partial}{\partial x^i}\right)_p, \left(\frac{\partial}{\partial x^j}\right)_p\right) \\ &= X^i(p)Y^j(p)g_{ij}(p) \end{aligned}$$

Where

$$g_{ij}(p) := g(p)\left(\left(\frac{\partial}{\partial x^i}\right)_p, \left(\frac{\partial}{\partial y^j}\right)_p\right)$$

We say g is C^∞ iff g_{ij} is $C^\infty, i, j = 1, \ldots, n$.

Change of variables

Let $\phi := \psi_2 \circ \psi_1^{-1}$ be an overlap map. Say

$$\begin{aligned} d\phi_p : \mathbb{R}^n &\to \mathbb{R}^n \\ \frac{\partial}{\partial x^i} &\mapsto \frac{\partial \phi^j}{\partial x^i}(x)\frac{\partial}{\partial y^j} \end{aligned}$$

or from another view point $\left(\frac{\partial}{\partial x^i}\right)_p = \frac{\partial \phi^j}{\partial x^i}(x)\left(\frac{\partial}{\partial y^j}\right)_p$ in T_pM. Then

$$\begin{aligned} g'_{ij}(x^1,\ldots,x^n) &= \langle\left(\tfrac{\partial}{\partial x^i}\right)_p, \left(\tfrac{\partial}{\partial x^j}\right)_p\rangle_p \\ &= \langle \tfrac{\partial \phi^k}{\partial x^i}(x)\left(\tfrac{\partial}{\partial y^k}\right)_p, \tfrac{\partial \phi^\ell}{\partial x^j}(x)\left(\tfrac{\partial}{\partial y^\ell}\right)_p\rangle_p \\ &= \tfrac{\partial \phi^k}{\partial x^i}(x^1,\ldots,x^n)\tfrac{\partial \phi^\ell}{\partial x^j}(x^1,\ldots,x^n) g_{k\ell}(y^1,\ldots,y^n) \end{aligned}$$

where $y^i = \phi^i(x^1,\ldots,x^n)$.
Briefly written: $g'_{ij} = \frac{\partial \phi^k}{\partial x^i}\frac{\partial \phi^\ell}{\partial x^j} g_{k\ell}$ (Change of variables)
Consequence: If g is smooth in one coordinate system, then g is smooth in all other coordinate systems.
Some things we get from a metric:

$$|X|_p := \sqrt{\langle X, X\rangle_p}$$

- lengths and angles in T_pM
- lengths of paths
- distance
- volume
- covariant differentiation
- etc…

Prefered identification of $(T_pM)^*$ with T_pM.

Example (Poincaré ball model of hyberbolic space)

$$g_{ij}(x) := \frac{4\delta_{ij}}{(1-|x|^2_{\text{euc}})^2},\ x \in B^n_1$$

where δ_{ij} is the Euclidean metric

$$X^i\delta_{ij}Y^j = \sum_i X^iY^i$$

Let γ be the path

$$\gamma(t) := (0,t) \in B^2$$

Compute

$$\begin{aligned}
\dot\gamma(t) &= (0,1)\\
|\dot\gamma|_g^2 &= \langle \dot\gamma(t), \dot\gamma(t)\rangle_{g(\gamma(t))}\\
&= \frac{4\delta_{ij}\dot\gamma^i(t)\dot\gamma^j(t)}{(1-|\gamma(t)|_{euc}^2)^2}\\
&= \frac{4|\dot\gamma(t)|_{euc}^2}{(1-|\gamma(t)|_{euc}^2)^2}\\
&= \frac{4\cdot 1}{(1-t^2)^2}\\
|\dot\gamma(t)|_g &= \frac{2}{1-t^2}
\end{aligned}$$

$$L(\gamma) = \int_{t=0}^{t=1} |\dot\gamma(t)|\, dt = \int_{t=0}^{t=1} \frac{2}{1-t^2} dt = \infty$$

Then hyperbolicspace is

$$\mathbb{H}^n := (B_1^n, g_{ij})$$

Homogeneous[7], isotropic[8], constant curvature $K=-1$. It is the only space with these properties (up to isometry).

Exercise Find an isometry of $\mathbb{H}^2$ that takes $(0,0)$ to $(a,0)$.

Theorem 7.1 *Every smooth manifold that is a union of countably many coordinate charts can be given a Riemannian metric.*

Remark For manifolds, "union of countably many coordinate charts" $\Leftrightarrow$ 2nd countable.

Let $\mathrm{Sym}^2(V^*)$ be the symmetric bilinear forms T on V. $\mathrm{Sym}^2_+(V^*) := \{T \in \mathrm{Sym}^2(V^*) |\ (X,X) > 0\ \forall X \in T_pM\}$.

Proposition 7.2 *$Sym^2_+(V^*)$ is a* convex cone *in the vector space $Sym^2(V^*)$.*

[7]all points look the same

[8]all directions look the same

7.1 Pullbacks of Metrics

Suppose $f : M^n \to (N^p, g)$ is smooth. Define the *pullback of g by f*, on M via

$$\begin{aligned} f^*(g)_p : T_pM \times T_pM &\to \mathbb{R}, p \in M, \\ f^*(g)(p)(X,Y) &:= g(f(p))\left(df_p(X), df_p(Y)\right), \ X, Y \in T_pM. \end{aligned}$$

Remark concerning $f^*(g)$

- $f^*(g)_{ij}(x) = \frac{\partial f^k}{\partial x^i}(x)\frac{\partial f^\ell}{\partial x^j}(x) g_{k\ell}(f(x))$ (verify!)
- pullback is *always* defined (no bijectivity requirements, in contrast to the case of vectors)
- $f^*(g)$ is bilinear, symmetric, nonnegative
- $f^*(g)$ is positive definite $\Leftrightarrow$ df_p is injective (so: f immersion $\Rightarrow$ $f^*(g)$ is a Riemannian metric)
- If f is a diffeomorphism then $f^*(g)$ is a perfect copy of g.

Definition An *isometry* is a diffeomorphism

$$f : (M, g) \to (N, h)$$

such that $f^*(h) = g$.

Definition

$$\operatorname{Isom}((M,g)) := \{f : M \to M | f^*(g) = g \text{ and } f \text{ a diffeomorphism}\}$$

Example $\operatorname{Isom}((S^n, \text{round})) \cong \mathrm{O}(n)$

Example (Poincaré upper half-plane model of hyperbolic space) Set $H := \{z = x + iy \in \mathbb{C} | \Im z > 0\}$, $\hat{g}_{ij}(z) := \frac{\delta_{ij}}{y^2}$. We obtain a second defintion of hyperbolic space

$$\mathbb{H}^2 := (H, \hat{g}_{ij}).$$

Exercise i. Find an isometry from the upper half-plane model to the Poincaré disk model:

$$(H, \hat{g}) \to (B_1^2, g)$$

ii. Show that the orientation preserving isometries of $(H, \hat{g})$ are

$$z \mapsto \frac{az+b}{cz+d} \qquad ad-bc>0,\ a,b,c,d \in \mathbb{R}$$

iii. Show

$$\operatorname{Isom}((H,g)) \cong \mathrm{GL}_+(2,\mathbb{R})/\mathbb{R}\cdot \mathbb{1} \cong \mathrm{SL}(2,\mathbb{R})/\{\pm\mathbb{1}\} =: \mathrm{PSL}(2,\mathbb{R})$$

(real) projective special linear group

iv. Show $\mathbb{H}^2$ is homogeneous and isotropic, i.e.
homogenous: $\forall p,q \in \mathbb{H}^2\ \exists$ isometry $p \mapsto q$.
isotropic at p: $\forall X,Y \in T_p\mathbb{H}^2\ \exists$ isometry fixing p and taking $X \mapsto Y$

Definition An *isometric immersion* of (M,g) into (N,h) is an immersion $f: M \to N$ such that $f^*(h) = g$. We call $f^*(h)$ *the metric induced by the immersion.*

Example Let $M \subseteq (N,h)$, with

$$\begin{aligned} i: M &\to N \\ x &\mapsto x \end{aligned}$$

be the inclusion map. Then $i^*(h)$ is the same as the induced metric we defined weeks ago, namely

$$\langle X,Y\rangle_p^M := \langle X,Y\rangle_p^N \quad \forall p \in M, \forall X,Y \in T_pM$$

Theorem 7.3 (Nash Embedding Theorem (hard)) *(M^n,g) Riemannian manifold compact (union of countable many charts). Then* $\exists$ isometric embedding

$$(M,g) \xrightarrow{f} (\mathbb{R}^p, \delta)$$

for some large p. (Here δ is the the standard metric on $\mathbb{R}^p$.)

7.2 Metrics on Lie groups

Theorem 7.4 *Every Lie group possesses a* left-invariant *metric, i.e a metric g such that*

$$L_a^*(g) = g\ \forall a \in G$$

where (recall)

$$\begin{aligned} L_a: G &\to G \\ b &\mapsto ab. \end{aligned}$$

Proof Let $g(e)$ be any inner product on T_eG. Where $e \in G$ is the identity element.
Note:

$$\begin{aligned} L_a : G &\to G \\ e &\mapsto a \\ (dL_a)_e : T_eG &\to T_aG \end{aligned}$$

Copy $g(e)$ from T_eG to T_aG via $(dL_a)_e$: for $X, Y \in T_aG$, set

$$g(a)(X,Y) := g(e)\left((dL_a)_e^{-1}(X), (dL_a)_e^{-1}(Y)\right)$$

It is trivial to verify that g is invariant under left translation by *any* $L_b : G \to G,\ b \in G$. One checks that $L_b : G \to G$ is an isometry i.e. $(dL_b)_a : (T_aG, g(a)) \to (T_{ba}G, g(ba))$ is an isometry $\forall a \in G$.

□

Exercise Prove a left-invariant metric on a Lie group is smooth.

Theorem 7.5 *Every Lie group has at least one left-invariant metric.*

Exercise Show that the metric induced on SO(n) by the standard inclusion

$$\mathrm{SO}(n) \subseteq M^{n\times n}(\mathbb{R}) = \mathbb{R}^{n^2}$$

is both left and right invariant (=: *bi-invariant*). Note that $M^{n\times n}(\mathbb{R})$ gets the metric induced by the inner product

$$\langle A, B\rangle := \sum_{i,j} A_i^j B_i^j$$

Theorem 7.6 *Every compact Lie group has a bi-invariant metric*[9].

Example We already saw that

$$L_a, R_a : S^3 \to S^3$$

are isometries.

[9]Do Carmo p-46 prob 7, Lee p.46 prob 3-10,11,12

7.3 Volume and Intergrals

Given a metric g and some map $u : M \to \mathbb{R}$, let us define integration on M

$$\int u \, d\mu \equiv \int_M u(x) \, d\mu_g(x)$$

3 ways to define it

- volume n-form: a section of $C^\infty(\bigwedge^n T^*M)$, namely $\sqrt{\det g_{ij}} dx^1 \wedge \cdots \wedge dx^n$
  - has a sign
  - M must be orientable
  - requires *exterior algebra*[10] (k-forms)
- Hausdorff measure $\mathcal{H}^n$
  - valid in any metric space $\mathcal{H}^n$
  - valid for any $\alpha \in [0, \infty)$
  - requires measure theory
- define in charts

  $$\int_U f(x^1, \dots, x^n) \sqrt{\det g_{ij}(x)} dx^1 \dots dx^n$$

  easiest

Basic Formula in a Chart

Let $(U, g_{ij}) \subseteq \mathbb{R}^n$. Define

$$\int_U f \, d\mu_g := \int_U f(x) \sqrt{det g_{ij}(x)} dx^1 \dots dx^n \qquad (\dagger\dagger)$$

Definition

- $C_c^0(M) := \{\text{continuous functions } M \to \mathbb{R} \text{ with } \textit{compact support}\}$
- support of u: supp:= $\overline{\{x | u(x) \neq 0\}}$

[10] Differential Topology

Desired properties of integration

$$I_g : u \mapsto \int_M u\, d\mu_g$$

i. $I_g : C^0_c(M) \to \mathbb{R}$ linear (over $\mathbb{R}$)

ii. I_g *positive*, i.e. $u \geq 0 \Rightarrow I_g(u) \geq 0$.

iii. I_g agrees with the usual integral on flat $\mathbb{R}^n$.

iv. (*Change of Variables / Area formula*)
If $\phi : (M, g) \xrightarrow{\phi} (N, h)$ is C^1 and bijective then

$$\int_N u(y) d\mu_h(y) = \int_M u(\phi(x)) |J\phi(x)|_{g,h} d\mu_g(x)$$

for any $u \in C^0_c$. Here $|J\phi(x)|$ is the volume expansion factor (Jacobian determinant) from $(T_x M, g(x))$ to $(T_{\phi(x)}, g(x))$

Theorem 7.7 *There exsits a unique system of maps*

$$u \mapsto \int_M u\, d\mu_g$$

with properties (i)-(iv). They are given locally by formula (††).

Remark (for measure theory experts)
$I_g \overset{\text{Riesz Rep. Thm}}{\longleftrightarrow}$ Radon measure μ_g.
I_g is a linear functional satisfying (i), (ii) and $\left|\int u\, d\mu_g\right| \leq C(K)\text{supp}|u|$ for spt $u \subseteq K \subseteq M$, with K compact.
μ_g is called the *Riemannian volume measure of g.*

Definition of the Jacobian determinant Suppose we are given

$$L : (V, g) \to (W, h) \text{ linear}$$

(V, g) and (W, h) being inner product spaces. Define

$$|JL| \equiv |JL|_{g,h} := \sqrt{\det(L^T L)}$$

Where the transpose $L^T : W \to V$ is characterized by $g(v, L^T w) = h(Lv, w)$

Motivation

Suppose $L : V \to V$ is linear. Then $\det L \in \mathbb{R}$ is defined (independent of coordinates and metrics!) Where as if $L : V \to W$,then $\det L$ is *not* defined. We note that $L^T L : V \to V$ is symmetric with respect to the inner product g, i.e. $g(v_1, L^T L v_2) = g(L^T L v_1, v_2)$.

Lemma 7.8 (Singular value Decomposition) *For any $L : (V, g) \to (W, h)$ there exists an orthonormal basis $v_1, \ldots, v_n$ of V and orthonormal basis $w_1, \ldots, w_n$ of W with $\lambda_1, \ldots, \lambda_n \geq 0$[11] such that $Lv_i = \lambda_i w_i$.*

Proof Diagonalize $L^T L$:

$$L^T L v_i := \mu_i v_i, i = 1, \ldots, n$$

where $v_1, \ldots, v_n$ is an orthonormal basis of V.
Observe:

$$h(Lv_i, Lv_j) = g(L^T L v_i, v_j) = g(\mu_i v_i, v_j) = 0$$

So $Lv_1, \ldots, Lv_n$ is an *orthogonal* set in W.
Define

$$w_i = \begin{cases} \frac{Lv_i}{|Lv_i|} & Lv_i \neq 0 \\ \text{any completion to orthonormal basis} & Lv_i = 0 \end{cases}$$

$$\lambda_i := |Lv_i| \geq 0.$$

Then $w_1, \ldots, w_n$ orthonormal basis with respect to h, and

$$Lv_i = \lambda_i w_i,$$

as required.

□

Further: $L^T w_i = \lambda_i v_i$, so $\mu_i = \lambda_i^2$. Thus

$$|JL|_{g,h} := \sqrt{\det(L^T L)} = \sqrt{\mu_1 \cdots \mu_n} = \lambda_1 \cdots \lambda_n$$

is seen to be the volume expansion factor of L from g to h.

[11] principal stretches

Definition Suppose $\phi : (M, g) \to (N, h)$ is C^1. Define

$$|J\phi(x)|_{g,h} := |Jd\phi(x)|_{g(x),h(\phi(x))}.$$

In coordinates: on V, W respectively, we have

$$g = (g_{ij}),\ h = (h_{kl}),\ L = (L_i^k),$$

and

$$\begin{array}{ccc} v \in V & \underset{L^T}{\overset{L=(L_i^k)}{\rightleftarrows}} & W \ni w \\ {\scriptstyle g^{-1}=(g^{ij})}\uparrow & & \downarrow{\scriptstyle h=(h_{ij})} \\ \nu \in V^* & \underset{L^*=(L_i^k)}{\longleftarrow} & W^* \ni \omega \end{array}$$

$h : W \to W^*$ is defined by

$$h(w) := h(w, \cdot) \in W^*$$

$g^{-1} : V^* \to V$ is characterized by

$$g(g^{-1}(\nu), \cdot) = \nu \in V^*$$

We find that $g^{-1} = (g^{ij})$, i.e. the matrix of the inverse of g is the inverse of the matrix of g. The dual map to L is defined by $L^*(\omega) := \omega \circ L \in L^*$. We have

$$v \mapsto Lv, (Lv)^k = L_i^k v^i$$

And also

$$\begin{array}{rcl} \omega & \mapsto & L^*\omega \\ (L^*\omega)_i & = & L_i^k \omega_k. \end{array}$$

To see the symmetry of this, observe

$$v^i L_i^k \omega_k = w(Lv) = (L^*(\omega))(v).$$

Next, we can verify

$$\begin{array}{rcl} L^T & = & g^{-1} \circ L^* \circ h, \\ (L^T)_\ell^i & = & g^{ij} L_j^k h_{k\ell} \end{array}$$

Formulae

- $$\begin{aligned} |J\phi(x)| &= \sqrt{\det(d\phi(x)^T \circ d\phi(x))} \\ &= \sqrt{\det(g^{ij}(x)\frac{\partial \phi^k}{\partial x^j}(x) h_{k\ell}(\phi(x))\frac{\partial \phi^\ell}{\partial x^i}(x))} \end{aligned}$$

- $|J\phi|_{\delta,\delta} = |\det(\frac{\partial \phi^i}{\partial x^j})| \overset{\phi(x)=x}{=} |J\phi|_{g,g}$
- $|J_{\text{id}}|_{\delta,g} = \sqrt{\det g_{ij}}$, if $\phi(x) = x$.
- $|J(\phi \circ \psi)|_{g,k} = |J\phi|_{h,k}|J\psi|_{g,h}$, where $(M,g) \xrightarrow{\psi} (N,h) \xrightarrow{\phi} (P,k)$

Local Formula

$$\int_U u\, d\mu := \int_U u(x) \underbrace{\sqrt{\det g_{ij}(x)}}_{J_{\text{id}}|_{\delta,g}} dx^1 \cdots dx^n \tag{‡}$$

Verify the Area Formula (in a chart)

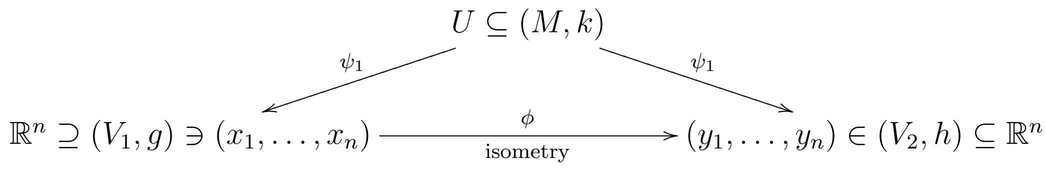
Given $\phi : (U,g) \to (V,h)$, C^1 and bijective with coordinates $x^1, \ldots, x^n$, $y^1, \ldots, y^n$ respectively. Show $\int_V u\, d\mu_h = \int_U u \circ \phi |J\phi|_{g,h}\, d\mu_g$.
Compute:

$$\begin{aligned} \text{LHS} &= \int_V u\sqrt{\det h_{k\ell}}\, dy^1 \cdots dy^n \\ &= \int_U u \circ \phi \sqrt{\det h_{k\ell} \circ \phi} \left| \det\left(\frac{\partial \phi^k}{\partial x^i}\right)\right| dx^1 \cdots dx^n \end{aligned}$$

(by the usual change of variables formula), where as

$$\text{RHS} = \int_U u \circ \phi \sqrt{\det\left(g^{ij}\frac{\partial \phi^k}{\partial x^j} h_{k\ell} \circ \phi \frac{\partial \phi^\ell}{\partial x^i}\right)} \sqrt{\det g_{ij}}\, dx^1 \cdots dx^n$$

Note By taking ϕ to be an *isometry*, this also verifies that our definition (‡) is independent of the coordinates that we chose on the open set $U \subseteq M$, as follows:



$g = (\psi_1)_*(k)$ $\qquad\qquad$ $h = (\psi_2)_*(k)$

Next step:

extend our defintion of the integral from each chart U to all of M. Say $M = \cup_\alpha U_\alpha$, then we must move from

$$\int_{U_\alpha} u \, d\mu_g \rightsquigarrow \int_M u \, d\mu_g$$

We obtain (as mentioned above)

Theorem 7.9 *There exists an integral $\int_M u \, d\mu_g$ that satisfies (i)-(iv)*

8 Connections

First we'll look at *connections on vector bundles* in general, then we'll specialize to the *Riemannian* or *Levi-Civita connection* on TM (induced by a Riemannian metric g)

8.1 Vector Bundles

(Lee Chap 2)
Let M be a smooth manifold. Attach a vector space E_p (disjoint!) to each point in M. Main example: $TM = \cup_p T_p M$.

Definition A *vector bundle of rank k over M* (base space) is a smooth manifold E (total space) together with a smooth map $\pi : E \to M$ such that

i. Each *fiber* $E_p := \pi^{-1}(p)$ is endowed with the structure of a k-dimensional vector space.

ii. For every $p \in M, \exists U \ni p$ open and a diffeomorphism

$$\Psi : \pi^{-1}(U) \to U \times \mathbb{R}^k$$

such that

iia. The following diagram commutes

$$\begin{array}{ccc} E \supseteq \pi^{-1}(U) & \xrightarrow{\Psi} & U \times \mathbb{R}^k \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_1} \\ M \supseteq U & = & U \end{array}$$

This says:

$$\Psi|E_p : E_p \to \{p\} \times \mathbb{R}^k$$

iib. $\Psi|E_p : E_p \to \{p\} \times \mathbb{R}^k$ is a linear isomorphism.

We call the map Ψ a *local trivialization (of E over U).* If U has coordinates $(x^1, \cdots x^n)$, then Ψ yields coordinates $(x^1, \ldots, x^n, \underbrace{V^1, \ldots, V^k}_{\text{coords on } \mathbb{R}^k})$ on $\pi^{-1}(U)$

Examples

TM

$T^*M := \cup_{p \in M} (T_pM)^*$ *cotangent bundle of M*

$M \times \mathbb{R}^k \xrightarrow{\pi} M$ *trivial bundle* (of rank k)

Simplest nontrivial vector bundle
$M = S^1$, Fiber$= \mathbb{R}$ (rank 1) Where

$$S^1 = [0, 2\pi]/(0 \sim 2\pi)$$

$$E := [0, 2\pi] \times \mathbb{R}/ \sim \ni (\theta, t),$$

where $(0, t) \sim (2\pi, -t)$

$$\begin{array}{rcl} \pi([\theta, t]) & = & [\theta] \\ \pi : E & \to & S^1 \end{array}$$

E is the Möbius band, viewed as a line bundle over S^1 We call it the *twisted $\mathbb{R}$-Bundle over S^1.*

Example

$$\cup_{p \in M} \text{Bilin}(T_pM \times T_pM \to \mathbb{R})$$

is a vector bundle over M of rank $k = n^2$. A metric is a smooth and positive section[12] of this bundle

[12] will be defined later

$\mathbb{R}^2$ bundles over S^2

$$\begin{array}{ccc} \mathbb{R}^2 & \longrightarrow & E \\ & & \downarrow \\ & & S^2 \end{array}$$

Give S^2 the "charts"
$H_+ :=$ closed northern hemisphere
$H_- :=$ closed southern hemisphere
$H_+ \cap H_- = \{\text{equator}\} \cong S^1$
To get S^2: glue H_+ to H_- along $\partial H_+, \partial H_-$ by the map

$$\begin{array}{rcl} \phi : \partial H_+ & \to & \partial H_- \\ e^{i\theta} & \mapsto & e^{i\theta} \end{array}$$

To get E: observe

$$\begin{array}{rcccl} \partial(H_+ \times \mathbb{R}^2) & = & (\partial H_+) \times \mathbb{R}^2 & \cong & S^1 \times \mathbb{R}^2 \\ \partial(H_- \times \mathbb{R}^2) & = & (\partial H_-) \times \mathbb{R}^2 & \cong & S^1 \times \mathbb{R}^2. \end{array}$$

Glue $H_+ \times \mathbb{R}^2$ to $H_- \times \mathbb{R}^2$ along their boundaries via

$$\Phi : \partial H_+ \times \mathbb{R}^2 \to \partial H_- \times \mathbb{R}^2$$

defined by

$$\Phi\left(e^{i\theta}, \begin{pmatrix} x \\ y \end{pmatrix}\right) := \left(\phi(e^{i\theta}), A_{e^{i\theta}} \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

Where we choose any family of linear maps

$$A_{e^{i\theta}} : \mathbb{R}^2 \to \mathbb{R}^2$$
$$A_{e^{i\theta}} \in \mathrm{GL}(2, \mathbb{R})$$
$$A : \partial H_+ \to \mathrm{GL}(2, \mathbb{R})$$

Our special choice: Fix $k \in \mathbb{Z}$, define

$$A : \partial H_+ \mapsto \mathrm{SO}(2) \subseteq \mathrm{GL}(2, \mathbb{R})$$

by

$$A(e^{i\theta}) := \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}.$$

We obtain

$$\Phi\left(e^{i\theta}, \begin{pmatrix} x \\ y \end{pmatrix}\right) := \left(e^{i\theta}, \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}\right)$$

The result is called the *k-twisted* $\mathbb{R}^2$ *bundle over* S^2

Question

What is k for the tangent bundle TS^2 of the 2-Sphere?

8.1.1 Complex vector bundles

Same definition, exept each E_p is a *complex* vector space of complex dimension d. Then $\dim_{\mathbb{R}} = n + 2d$[13].

Question

Can you think of a real vector bundle of even rank that *cannot* be made into a complex vector bundle?

Definition Let $M \xrightarrow{f} N$ with vector bundles E and F over M and N respectivly. A *(linear) bundle map over* f is a smooth map

$$L : E \to F$$

such that

$$\begin{array}{ccc} E & \xrightarrow{L} & F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{f} & N \end{array}$$

commutes, i.e.$L(E_p) \subseteq F_{f(p)}$ and

$$L_p := L|E_p : E_p \to F_{f(p)}$$

is linear map.

Definition A *bundle isomorphism* is a (linear) bundle map that is a diffeomorphism[14]

Example In an exerciese, we found a bundle isomorphism

$$\begin{array}{ccc} TS^3 & \xrightarrow[\cong]{} & S^3 \times \mathbb{R}^3 \ ^{17} \\ \downarrow & & \downarrow \\ S^3 & = & S^3 \end{array}$$

$i, j, k \in C^\infty(TS^3)$ and $i(p), j(p), k(p)$ form a basis for $T_pS^3 \, \forall p$

$$(p, (x, y, z)) \mapsto (p, xi(p) + yj(p) + zk(p))$$

[13] as a real manifold

[14] check: this is equivalent to: f is a diffeomorphism and $L|E_p$ is a linear isomorphism $\forall p$.

[17] Trivial bundle, $p \in S^3, (x, y, z) \in \mathbb{R}^3$

Definition A *subbundle* of E is a submanifold $F \subseteq E$ such that $F_p := F \cap E_p (= (\pi|F)^{-1}(p))$ is a vector subspace of E (of constant dimension). F is then (check!) a vector bundle over M in it's own right.

$$\begin{array}{ccc} F & \subseteq & E \\ {\scriptstyle \pi|F}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & = & M \end{array}$$

Example

i. $M^n \subseteq \mathbb{R}^q$ submanifold TM[18] $= \cup\{p\} \times T_pM \subseteq M \times \mathbb{R}^q$[19] is a subbundle with $n \leq q$.

ii.
$$NM := \cup_{p \in M}\{p\} \times N_pM \subseteq M \times \mathbb{R}^q$$
subundle (called *normal bundle of M in* $\mathbb{R}^q$, $N_pM = (T_pM)^{\perp}$).

Definition A *section* of E is a function $V : M \to E$ such that $V(p) \in E_p, p \in M$. We call V *smooth* if it is smooth as a map between smooth manifolds.

Definition The *0-section* is the section $O(p) := 0 \in T_pM, p \in M$.

$\Gamma(E)$: all sections
$C^\infty(E)$: all smooth sections
Both of the above are vector spaces over $\mathbb{R}$

$$V, W \in C^\infty(E) \Rightarrow aV + bW \in C^\infty(E)$$

Definition A *local frame for E* is a list $e_1(p), \ldots, e_d(p), p \in U$ of sections in $C^\infty(E|U)$ that form a basis for E_p at each $p \in U$.
A local fram alway yields a local trivialization (and viceversa)

Given a frame over U, we may express any section V locally as a linear combination:
$$V(p) = V^\alpha(p) e_\alpha(p), p \in U$$
Where V^α are the component functions
Evidently: V is smooth iff each component function V^α is smooth. Thus $v, w \in C^\infty(E) \Rightarrow aV + bW \in C^\infty(E)$.

[18] rank n
[19] trivial bundle over M with fiber $\mathbb{R}^q$ (rank q).

Example

$$\text{Bilin}(TM, TM; \mathbb{R}) := \cup_{p \in M} \text{Bilin}(T_pM \times T_pM \to \mathbb{R})$$

can be given the structure of a smooth vector bundle over M, and a Riemannian metric is a (smooth, symmetric, positive) section of this bundle.

Example Every smooth section of the twisted $\mathbb{R}$-bundle over S^1 has a zero

8.2 Connections on Vector Bundles

Aim: Given $\tilde{X} \in T_pM, V \in C^\infty(E)$, form

$$\mathcal{D}_{\tilde{X}} V \in E_p$$

directional derivative of V *in the direction* $\tilde{X}$ *at* p.

[Try:]

- $X^i \frac{\partial V^\alpha}{\partial x^i}$, $X = X^j \frac{\partial}{\partial x^j}, V = V^\alpha e_\alpha$.
  Does not transform correctly (depends on choice of frame).
- $\frac{d}{dt}\Big|_{t=0} \frac{V(\gamma(t)) - V(\gamma(0))}{t}$ where γ is a path in M, $\gamma(0) = p, \dot{\gamma}(0) = \tilde{X}$.
  Cannot compare vectors in $E_\gamma(t)$ to $E_{\gamma(0)}$ in an intrinsic way.

Upshot To differntiate V in directions $\tilde{X}$, we must *declare*, or *impose* a structure E called a connection

Definition

$$E \to M \quad \text{vector bundle}$$

An *(affine) connection* or *covariant derivative operator*, on E is a map

$$\begin{array}{cccccc} \mathcal{D} : C^\infty(TM) & \times & C^\infty(E) & \to & C^\infty(E) \\ X & & V & \mapsto & \mathcal{D}_X V \end{array}$$

that satisfies

- $\mathcal{D}_X(aV + bW) = a\mathcal{D}_X V + b\mathcal{D}_X W, a, b \in \mathbb{R}$ (linear in V over $\mathbb{R}$)
- $\mathcal{D}_{fX+gY} V = f\mathcal{D}_X V + g\mathcal{D}_Y V, f, g \in C^\infty(M)$ (linear in X over $C^\infty(M)$)
- $\mathcal{D}_X(fV) = f\mathcal{D}_X V + (X \cdot f)V, f \in C^\infty(M)$ (Leibniz rule)

Expression in coordinates

$X = X^i \frac{\partial}{\partial x^i}, V = V^\alpha e_\alpha$ over U

$$\begin{aligned} \mathcal{D}_X V &= \mathcal{D}_{X^i \frac{\partial}{\partial x^i}} (V^\alpha e_\alpha) \\ &= X^i \mathcal{D}_{\frac{\partial}{\partial x^i}} (V^\alpha e_\alpha) \\ &= X^i \left((\tfrac{\partial}{\partial x^i} \cdot V^\alpha) e_\alpha + V^\alpha \mathcal{D}_{\frac{\partial}{\partial x^i}} e_\alpha \right) \end{aligned}$$

Definition The *connection coefficients* are defined by

$$\left(\mathcal{D}_{\frac{\partial}{\partial x^i}} e_\alpha\right)_p = \Delta^\beta_{i\alpha}(p) e_\beta(p)^{20}, \; p \in U \; i = 1, \dots, n, \; \alpha = 1, \dots, d$$

$$\Delta^\beta_{i\alpha} = \Delta^\beta_{i\alpha}(p), \Delta^\beta_{i\alpha} \in C^\infty(U)$$

Get:

$$\mathcal{D}_X V = X^i \frac{\partial V^\alpha}{\partial x^i} e_\alpha + X^i V^\alpha \Delta^\beta_{i\alpha} e_\beta$$

or, writing $\mathcal{D}_X V = (\mathcal{D}_X V)^\alpha e_\alpha$:

$$\boxed{(\mathcal{D}_X V)^\alpha = X^i \tfrac{\partial V^\alpha}{\partial x^i} + X^i V^\beta \Delta^\alpha_{i\beta}}$$

i.e. derivative plus correction term.

This shows:

- $\mathcal{D}_X V(p)$ dependas linearly on the value of V and it's first derivatives at p.
- $\mathcal{D}_X V(p)$ depends linearly only on $X(p)$ and not on any derivatives of X. We say $\mathcal{D}_X V$ is *tensorial in* X or *point wise in* X.

As a result, we may define

$$\mathcal{D}_{\tilde{X}} V, \tilde{X} \in T_p M, V \in C^\infty(E)$$

via

$$\mathcal{D}_{\tilde{X}} V := \mathcal{D}_X V(p)$$

where $X \in C^\infty(TM)$ is any vectorfield such that $X(p) = \tilde{X}$.
This yields a linear map

$$\begin{aligned} \mathcal{D}V(p) : T_p M &\rightarrow E_p \\ \tilde{X} &\mapsto \mathcal{D}_{\tilde{X}} V \\ (\mathcal{D}V(p))(\tilde{X}) &\equiv \mathcal{D}_{\tilde{X}} V \end{aligned}$$

[20] nd^2 functions on U

$$\mathcal{D}V(p) \in \mathrm{Hom}(T_pM, E_p)$$

We can form a vector bundle

$$\begin{aligned} \mathrm{Hom}(TM, E) &:= \cup_{p\in M}\mathrm{Hom}(T_pM, E_p) \\ \mathcal{D}V &:= (\mathcal{D}V(p))_{p\in M} \in C^\infty(\mathrm{Hom}(TM, E)) \end{aligned}$$

More comments on the formula:

$$(\mathcal{D}_X V)^\alpha = X^i \frac{\partial V^\alpha}{\partial x^i} + X^i V^\beta \Delta^\alpha_{i\beta}$$

$X^i \frac{\partial V^\alpha}{\partial x^i}$ defines the connection
$\mathcal{D}^0_X V := X^i \frac{\partial V^\alpha}{\partial x^i} e_\alpha$ defines a connection (check!) called the *coordinate connection* induced by the frame $e_1, \ldots, e_d$, $d \equiv \mathrm{rank} E$.
So $\mathcal{D}^0$ has the property: $\mathcal{D}^0_X e_\alpha = 0 \ \forall X \in C^\infty(TM)$.

Definition We call a section $V \in C^\infty(E)$ *parallel (for $\mathcal{D}$)* if $\mathcal{D}_X V = 0 \ \forall X \in C^\infty(TM)$.

Example $\mathbb{R}^n, E = T\mathbb{R}^n, e_i \equiv \frac{\partial}{\partial x^i}$

$$\left(\mathcal{D}^0_X Y\right)^j = X^i \frac{\partial Y^j}{\partial x^i}$$

(usual directional derivative)
Y *parallel* iff components are *constant*

Remark It is rare for a connection to have even *one* parallel section.

Exercise For any choice of nd^2 smooth functions $\Delta^\beta_{i\alpha}, p \in U$, the above formula yields a connection.

The correction term yields a bilinear map

$$\tilde{X}, \tilde{V} \mapsto \tilde{X}^i \tilde{V}^\beta \Delta^\alpha_{i\beta}(p) e_\alpha(p) \in E_p$$

$$\tilde{X} \in T_pM, \tilde{V} \in E_p$$

to which we give the name

$$\Delta(p) : T_pM \times E_p \to E_p$$

So $\Delta(p) \in \mathrm{Bilin}(T_pM, E_p; E_p)$. We form a smooth vector bundle

$$\mathrm{Bilin}(TM, E; E) := \cup_{p\in M}\mathrm{Bilin}(T_pM, E_p; E_p)$$

and we recognize that

$$\Delta := (\Delta(p))_{p\in M} \in C^\infty(\text{Bilin}(TM, E; E))$$

$$\Delta : M \to \text{Bilin}(TM, E; E), p \mapsto \Delta(p)$$

Define

$$\Delta(X, V) \in C^\infty(E)$$

$$\Delta(X, V)(p) := \Delta(p)\,(X(p), V(p))$$

$$\Delta : C^\infty(TM) \times C^\infty(E) \to C^\infty(E)$$

So we can write:

$$\mathcal{D}_X V = D^0_X V + \Delta(X, V)$$

$$\mathcal{D} = \mathcal{D}^0 + \Delta$$

Theorem 8.1

i. *The difference between any two connections on E yields a section of Bilin$(TM, E; E)$.*

ii. *Any connection plus any smooth section of Bilin$(TM, E; E)$ yields another connection.*

Example

$$\begin{array}{c} E = S^1 \times \mathbb{R} \ni (\theta, t) \\ \downarrow \\ M = S^1 \end{array}$$

$e_1(\theta) = (\theta, 1)$

$$V \in C^\infty(E),\ V(\theta) = V^1 e_1(\theta),\ \Delta^1_{11} = a(\theta)$$

$$X = \frac{\partial}{\partial\theta},\ \mathcal{D}_{\frac{\partial}{\partial\theta}} V = \frac{\partial V^1}{\partial\theta} e_1 + a(\theta) V^1(\theta) e_1$$

Let $a(\theta) = -\frac{1}{10}$

$$\mathcal{D}_{\frac{\partial}{\partial\theta}} V = \frac{\partial V^1}{\partial\theta} e_1 - \frac{1}{10} V^1 e_1$$

Equation for parallel section:

$$0 = \left(\frac{\partial V^1}{\partial \theta} - \frac{1}{10}V^1\right) e_1$$

$$\frac{dV^1}{d\theta} = \frac{1}{10}V^1$$

$$V^1(\theta) = ce^{\theta/10}, c = 1$$

This connection has no (global) parallel section.

$$\mathcal{D}_{\frac{\partial}{\partial\theta}} e_1 = -\frac{1}{10}e_1$$

i.e. $e_1(\theta)$ is decreasing in length (compared to a parallel section) at rate $-\frac{1}{10}e_1$.

8.3 Inner Products on E and compatible connections

$$(E, \langle\cdot,\cdot\rangle) \quad \text{Euclidean bundle}$$

Suppose we have $\langle\cdot,\cdot\rangle_p : E_p \times E_p \to \mathbb{R}, p \in M$ a smooth family of inner products on the fibers of E.

Definition $\mathcal{D}$ is *compatible* with $\langle\cdot,\cdot\rangle$ if

$$X \cdot \langle V, W\rangle = \langle \mathcal{D}_X V, W\rangle + \langle V, \mathcal{D}_X W\rangle \ \forall X \in C^\infty(TM), V, W \in C^\infty(E)$$

$$\text{(Leibniz rule)} \quad X \cdot |V|^2 \ = \ \langle \mathcal{D}_X V, V\rangle + \langle V, \mathcal{D}_X V\rangle$$

Exercise

i. Prove if $\mathcal{D}$ is compatible with $\langle\cdot,\cdot\rangle$, and V is parallel for $\mathcal{D}$, then $|V|^2$ is constant on M if M is connected.

ii. Show the connection

$$\mathcal{D}_{\frac{\partial}{\partial\theta}} V = \left(\frac{\partial V^1}{\partial \theta} - \frac{1}{10}V^1\right) e_1$$

is not compatible with *any* inner product.

8.4 Riemannian Connections

Also called *Levi-Civita Connection of a metric* g. $M, g \rightsquigarrow \mathcal{D} = \mathcal{D}^g$ on TM.

Definition A connection $\mathcal{D}$ on TM is called *torsion-free* or symmetric if

$$\mathcal{D}_X Y - \mathcal{D}_Y X = [X, Y] \ \forall X, Y \in C^\infty(TM). \qquad (\odot)$$

Example

- True for the usual directional derivative in $\mathbb{R}^n$

$$[X, Y]^j = X^i \frac{\partial Y^j}{\partial x^i} - Y^i \frac{\partial X^j}{\partial x^i}$$

- all coordinate connections on TM are torsion free.

Interpretation of $\odot$

The antisymmetric part of $\mathcal{D}_X Y$ is given by something that comes from the smooth structure alone. $[X, Y]$.
In particular:

$$\mathcal{D}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \mathcal{D}_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^i}$$

(since $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$)

Theorem 8.2 *For every* (M, g) *there exists a unique connection on* TM *that is*

- *symmetric*
- *compatible with* g

In coordinates:

$$\mathcal{D}_X Y = X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial}{\partial x^j} + X^i Y^j \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

where

$$\mathcal{D}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k} \text{ (defines } \Gamma^k_{ij}(p).)$$

Then $\mathcal{D}$ is symmetric iff $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Proof Symmetry in coordinates:

$$\left(X^i\frac{\partial Y^k}{\partial x^i}+X^iY^j\Gamma^k_{ij}\right)-\left(Y^i\frac{\partial X^j}{\partial x^i}+Y^iX^j\Gamma^k_{ij}\right)$$

$$\begin{aligned} &= X^i\frac{\partial Y^k}{\partial x^i}-Y^i\frac{\partial X^k}{\partial x^i} \\ X^iY^j\Gamma^k_{ij} &= Y^iX^j\Gamma^k_{ij}\ \forall X,Y \\ &\Leftrightarrow \Gamma^k_{ij}=\Gamma^k_{ji} \end{aligned}$$

□

Theorem 8.3 (Levi-Civita) *Given (M,g), there exists a unique connection $\mathcal{D}$ on TM satisfying*

i. $\mathcal{D}$ is compatible with g

ii. $\mathcal{D}$ is torsion-free

$\mathcal{D}$ is called the Levi-Civita or Riemannian connection of g.

Proof of uniqueness

$$\begin{aligned} X\cdot\langle Y,Z\rangle &= \langle D_XY,Z\rangle+\langle Y,D_XZ\rangle \\ Y\cdot\langle Z,X\rangle &= \langle D_YZ,X\rangle+\langle Z,D_YX\rangle \\ Z\cdot\langle X;Y\rangle &= \langle D_ZX,Y\rangle+\langle X,D_ZY\rangle \end{aligned}$$

$$\begin{aligned} &X\cdot\langle Y,Z\rangle+Y\cdot\langle Z,X\rangle-Z\cdot\langle X,Y\rangle \\ &=\langle [Y,Z],X\rangle+\langle [X,Z],Y\rangle-\langle [X,Y],Z\rangle+2\langle D_xY,Z\rangle \Rightarrow \text{uniqueness} \end{aligned}$$

$$\begin{aligned} \langle D_XY,Z\rangle =& \frac{1}{2}\left(X\cdot\langle Y,Z\rangle+Y\cdot\langle X,Z\rangle-Z\cdot\langle X,Y\rangle\right. \\ & \left.-\langle Y,[X,Z]\rangle-\langle X,[Y,Z]\rangle+\langle Z,[X,Y]\rangle\right) \end{aligned} \tag{‡}$$

- uniquely characterizes $\mathcal{D}_XY$ in terms of g and smooth structure of M.
- not quite a formula for $\mathcal{D}_XY$ (derivatives of Z appear on right hand side).

Find a formula for $\mathcal{D}_X Y$
Insert $X = \frac{\partial}{\partial x^i}, Y = \frac{\partial}{\partial x^j}, Z = \frac{\partial}{\partial x^k}$, $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$. Recall $g_{ij} = \langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle$

$$\langle \underbrace{\mathcal{D}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}}_{\Gamma_{ij}^m \frac{\partial}{\partial x^m}}, \frac{\partial}{\partial x^k} \rangle = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right)$$

Recall

$$(\mathcal{D}_X Y)^k = X^i \frac{\partial Y}{\partial x^i} + \Gamma_{ij}^k X^i Y^j$$

where $\Gamma_{ij}^k \frac{\partial}{\partial x^k} = \mathcal{D}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$ defines Γ_{ij}^k.

$$\begin{aligned} \text{LHS} &= \langle \Gamma_{ij}^m \frac{\partial}{\partial x^m}, \frac{\partial}{\partial x^k} \rangle \\ &= \Gamma_{ij}^m g_{mk} = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right) \end{aligned}$$

multiply by g^{-1}=(g^{kl})
Get:

$$\boxed{\Gamma_{ij}^\ell = \frac{1}{2} g^{\ell k} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right)} \qquad (\dagger\ddagger)$$

classic formula for *Christoffel symbols* Γ_{ij}^k.
Where

$$(\mathcal{D}_X Y)^\ell = X^i \frac{\partial Y^\ell}{\partial x^i} + X^i Y^j \Gamma_{ij}^\ell \qquad (\#)$$

Formulas ($\dagger\ddagger$) and ($\#$) define a differntial operator $\mathcal{D}$.
It remains to verify (existence part of theorem)

- $\mathcal{D}$ is a connection (previous exercise)
- $\mathcal{D}$ is symmetric (because $\Gamma_{ij}^k = \Gamma_{ji}^k$)
- $\mathcal{D}$ is compatible with g.

Must verify:

$$X \cdot \langle Y, Z \rangle = \langle \mathcal{D}_X Y, Z \rangle + \langle Y, \mathcal{D}_X Z \rangle$$

In coordinates:

$$\begin{aligned} X^i \frac{\partial}{\partial x^i} \left(Y^j Z^k g_{jk} \right) \stackrel{?}{=} & \left(X^i \frac{\partial Y^\ell}{\partial x^i} + X^i Y^j \Gamma_{ij}^\ell \right) g_{\ell k} Z^k \\ & + \left(X^i \frac{\partial Z^\ell}{\partial x^i} + X^i Z^k \Gamma_{ik}^\ell \right) g_{\ell j} Y^j \end{aligned}$$

$$X^i\left(\frac{\partial Y^j}{\partial x^i}Z^k g_{jk}+Y^j\frac{\partial Z^k}{\partial x^i}g_{ik}+Y^jZ^k\frac{\partial g_{jk}}{\partial x^i}\right)\Leftrightarrow\frac{\partial g_{jk}}{\partial x^i}\stackrel{?}{=}\Gamma^\ell_{ij}g_{\ell k}+\Gamma^\ell_{ik}g_{\ell j}$$

This last statement is true, as seen by substitution.

□

8.5 Parallel Transport

parallel transport of a vector around a 90-90-90 triangle in S^2 creates a 90 rotation.
$E \to M$ bundle, $\gamma : [a,b] \to M$ smooth curve. ($E = TM$: main example).

Definition A *(smooth) section of* E *along* γ is a smooth function $V : [a,b] \to E, V(t) \in E_{\gamma(t)}$ $\forall t \in [a,b]$

Allowed:

- self-intersections
- $\dot\gamma = 0$

Wish to make sense of "$\mathcal{D}_{\dot\gamma}V$"

$$(\mathcal{D}_{\dot\gamma}\tilde V)^\alpha = \underbrace{\dot\gamma^i\frac{\partial\tilde V^\alpha}{\partial x^i}}_{\frac{dV^\alpha}{dt}} + \dot\gamma^i\tilde V^\beta\Delta^\alpha_{i\beta},\ \tilde V \in C^\infty(E)$$

$e_\alpha(x)$ local frame for E

$$V(t) = V^\alpha(t)e_\alpha(\gamma(t))$$

Notation

$$\frac{\mathcal{D}V}{dt} := \left(\frac{dV^\alpha(t)}{dt} + \dot\gamma^i(t)V^\beta(t)\Delta^\alpha_{i\beta}(\gamma(t))\right)e_\alpha(\gamma(t))$$

"$\mathcal{D}_{\dot\gamma}V$" *covariant derivative of* V *along* γ

Clearly

- $\frac{\mathcal{D}V}{dt}$ is a smooth section of E along γ
- $\frac{\mathcal{D}(fV)}{dt} = \frac{df}{dt}V + f\frac{\mathcal{D}V}{dt}$, $f = f(t)$

- $\frac{d}{dt}\langle V,W\rangle = \langle\frac{\mathcal{D}V}{dt},W\rangle + \langle V,\frac{\mathcal{D}W}{dt}\rangle$ if $\mathcal{D}$ is compatible with some inner product $\langle\cdot,\cdot\rangle$ on E.
- If V is obtained from an ambient section $\tilde{V} \in C^\infty(E|U)$ ($U \supseteq \mathrm{Im}\gamma$) (open) via $V(t) = \tilde{V}(\gamma(t))$ then $\frac{\mathcal{D}V}{dt}(t) = \mathcal{D}_{\dot\gamma}\tilde{V}$

Definition A section V along γ is called *parallel along* γ if $\frac{\mathcal{D}V}{dt} = 0\ \forall t \in [a,b]$.

Proposition 8.4 *Fix* $\gamma : [a,b] \to M, \tilde{V} \in E_a$. *Then there exists a unique parallel section* $V(t)$ *along* γ *such that* $V(a) = \tilde{V}$.

Proof In a fixed chart U we may solve the $d \times d$ system of ODES that says $\frac{\mathcal{D}V}{dt} = 0, \hat{V}(a) = \tilde{V}$, namely

$$(*)\begin{cases} \frac{dV^\alpha(t)}{dt} + \dot\gamma^i(t)V^\beta(t)\Gamma^\alpha_{i\beta} = 0, & \alpha = 1,\dots,d \\ V^\alpha(a) = \hat{V}, & \alpha = 1,\dots,d \end{cases}$$

for smooth functions $V^1(t),\dots,V^d(t)$ $t \in [a,c]$, as long as $\gamma([a,c]) \subseteq U$. Now select $a = t_0 < t_1 < \cdots < t_s = b$ such that each $\gamma([t_i,t_{i+1}])$ lies in a single chart U_i. Existence follow by induction. Uniqueness, smoothness also follow from ODE theory.

□

Definition *Parallel transport* is defined along γ from $\gamma(a)$ to $\gamma(b)$ as the map

$$\begin{aligned} P_\gamma : E_{\gamma(a)} &\to E_{\gamma(b)} \\ \hat{V} = V(a) &\to V(b) \end{aligned}$$

P_γ is linear since the ODE system we solved to find $P_\gamma(\hat{V})$ is linear.

Proposition 8.5 *If* $\mathcal{D}$ *is compatible with* $\langle\cdot,\cdot\rangle$ *then* P_γ *is an* isometry *from* $E_{\gamma(a)}$ *to* $E_{\gamma(b)}$.

Proof Let $V(t), W(t)$ be parallel along γ. Then

$$\frac{d}{dt}\langle V,W\rangle = \langle\frac{\mathcal{D}V}{dt},W\rangle + \langle V,\frac{\mathcal{D}W}{dt}\rangle = 0 + 0$$

So $\langle V(t),W(t)\rangle$ is constant.

□

Example Let γ be a great circle (transversed at unit speed) on S^2. $\mathcal{D}^{S^2}$ is the Levi-Civita connection of the induced metric an S^2.

Claim $\dot{\gamma}$ is parallel along γ i.e. $\mathcal{D}^{S^2}_{\dot{\gamma}}\dot{\gamma} = 0$

Lemma 8.6 *(Proof will be an exercise) Given (M, g), and $N \subseteq M$ submanifold.*

$$\begin{array}{ccc} g & \xrightarrow[\text{to } N]{\text{restriction}} & h \\ \downarrow & & \downarrow \tfrac{1}{2}\,hour \\ \mathcal{D}^g & \overset{\text{orthogonal}}{\underset{\text{projection}}{\dashrightarrow}} & \mathcal{D}^h = \mathcal{D}' \end{array}$$

$$h_p(X, Y) := g_p(X, Y),\ \ p \in N, X, Y \in T_pN$$
$$\pi^{TN}(p) : T_pM \to T_pN$$

orthogonal projection.
Exercise X-I

$$D'_X Y := \pi^{TN}(\mathcal{D}^g_{\tilde{X}}\tilde{Y})$$

$\tilde{X}, \tilde{Y} \in C^\infty(TM)$ extend $X, Y \in C^\infty(TN)$. $\mathcal{D}'$ is a connection on TN. $(\tilde{X}|N = X, \tilde{Y}|N = Y)$

$$\mathcal{D}^M_{\tilde{X}}\tilde{Y} = \underbrace{\mathcal{D}^N_X Y}_{\textit{tangental part}} + \textit{normal part}$$

Proof of Claim Setup:

$$e_1 \perp e_2 \in \mathbb{R}^3, |e_1| = |e_2| = 1$$

$$\gamma(t) = \cos t e_1 + \sin t e_2$$
$$\dot{\gamma} = \frac{d\gamma}{dt} = -\sin t e_1 + \cos t e_2$$
$$\mathcal{D}^{\mathbb{R}^3}_{\dot{\gamma}}\dot{\gamma} = \frac{d^2\gamma}{dt^2} = -\cos t e_1 - \sin t e_2 = -\gamma$$

Calculate:

$$\begin{aligned} \mathcal{D}^{S^2}_{\dot{\gamma}}\dot{\gamma} &= \pi^{TS^2}(\mathcal{D}^{\mathbb{R}^3}_{\dot{\gamma}}\dot{\gamma}) \\ &= \pi^{TS^2}(-\gamma) \\ &= 0 \end{aligned}$$

□

Observe: a continuous vector field $V(t)$ is parallel along γ iff $|V(t)|^2$ is constant, $\langle V(t), \dot{\gamma}(t)\rangle$ is constant.

Example $S^2 \subseteq \mathbb{R}^3$ If β traverses a 90-90-90 trianlge in S^2, then

$$P_\beta : T_pM \to T_pM$$

is rotation by 90.

Definition If γ is a closed curve in M, $\gamma(a) = \gamma(b) = p, \mathcal{D}$ cannon $E \to M$, the linear map $P_\gamma : E_p \to E_p$ is called the *holonomy map*.

9 Geodesics, Exponential Map

A *geodesic* is a curve with zero acceleration this is equivalent to a locally length-minimizing curve. Define the acceleration (with respect to $\mathcal{D}$) as

$$\ddot{\gamma} := \frac{\mathcal{D}\dot{\gamma}}{dt} ='' \mathcal{D}_{\dot{\gamma}}\dot{\gamma}''$$

(a vector field along γ))

Definition γ is a *geodesic* if $\ddot{\gamma}(t) = 0$, $t \in [a, b]$. "Motion of a free particle in a Riemannian manifold".

Example A great circle of unit speed in S^n is a geodesic

Remarks

- $\frac{d}{dt}|\dot{\gamma}|^2 = 2\langle\ddot{\gamma}, \dot{\gamma}\rangle = 0$ so $|\dot{\gamma}|$ is constant (constant speed)
- Let $\gamma(t)$ be a geodesic $\Rightarrow \beta(t) := \gamma(ct)$ is a geodesic. $\dot{\beta} = c\dot{\gamma}, \ddot{\beta} = c^2\ddot{\gamma}$

ODE for geodesics

Coordinates $x^1, \ldots, x^n$ on $U \subseteq M$. Write

$$\begin{aligned}
\gamma(t) &= (\gamma^1(t), \ldots, \gamma^n(t)) \\
\dot{\gamma}^i(t) &= \frac{d\gamma^i}{dt}(t) \\
\ddot{\gamma}^i(t) &= \left(\frac{\mathcal{D}\dot{\gamma}}{dt}\right)^i(t) \\
&= \frac{d\dot{\gamma}^i}{dt} + \dot{\gamma}^j\dot{\gamma}^k\Gamma^i_{jk}(\gamma(t))
\end{aligned}$$

so γ is a geodesic iff

$$\frac{d^2\gamma^i}{dt^2} + \frac{d\gamma^j}{dt}\frac{d\gamma^k}{dt}\Gamma^i_{jk}(\gamma(t)) = 0, i = 1, \ldots, n \tag{1}$$

$n \times n$ system of nonlinear ODEs.(linear in 2nd order derivatives quadratic in 1st oder, fully nonlinear in γ itself.)
Consider the initial conditions

$$\begin{cases} \gamma(0) = p \\ \dot{\gamma}(0) = X \end{cases} \tag{2}$$

$p \in M, X \in T_pM$

Theorem 9.1 (Short-term existence for geodesics) *Forall $p \in M$ and all $X \in T_pM$ there is a unique solution $\gamma = \gamma_{p,X} : [0, \varepsilon) \to M$ of (1) and (2) for some $\varepsilon > 0$.*

Proof later

□

Definition The *exponential map* by

$$\exp_p : \{\text{subset of } T_pM\} \to M$$

by

$$\exp_p(X) := \gamma_{p,X}(1)$$

whenever this exists.

Lemma 9.2 (Homogeneity)

i. $\gamma_{p,sX}(t) = \gamma_{p,X}(st)$

ii. $t \mapsto \exp_p(tX)$ *is a geodesic.*

Proof

i. $t \mapsto \gamma_{p,X}(st)$ is a geodesic by the above remark, with $\frac{d}{dt}\big|_0 \gamma_{p,X}(st) = s\,\frac{d}{dt}\big|_0 \gamma_{p,X}(t) = sX$ so $t \mapsto \gamma_{p,X}(st)$ and $t \mapsto \gamma_{p,sX}(t)$ have the same initial point, and the same initial velocity so by uniqueness of geodesics they are the same

ii.

$$\begin{aligned} \exp_p(tX) &= \gamma_{p,tX}(1) \\ &\overset{1}{=} \gamma_{p,X}(t) \end{aligned}$$

which is a geodesic.

□

9.1 Geodesic Flow

Rewrite (1),(2) (equations and initial conditions for geodesics) as a $2n \times 2n$ 1st order ODE system for $(\gamma^1(t), \ldots, \gamma^n(t), Y^1(t), \ldots, Y^n(t)) \in TM$ where M has the coordinates $(x^1, \ldots, x^n, X^1, \ldots, X^n)$ and $Y^i(t)$ shall end up being $\frac{d\gamma^i}{dt}(t)$.
Get:

$$\begin{cases} \frac{d\gamma^i}{dt} &= Y^i(t), & i = 1, \ldots, n \\ \frac{dY^i}{dt} &= -Y^p(t)Y^q(t)\Gamma^i_{pq}(\gamma(t)), & i = 1, \ldots, n \end{cases} \tag{1'}$$

$$\gamma(0) = p, \ Y(0) = X \tag{2'}$$

Rewrite as

$$\frac{d\tilde{\gamma}}{dt} = G(\tilde{\gamma}) \tag{1''}$$

$$\tilde{\gamma}(0) = (p, X) \tag{2''}$$

where

$$\tilde{\gamma}(t) = (\gamma(t), Y(t)) \ Y(t) = Y^i(t) \left(\frac{\partial}{\partial x^i}\right)_{\gamma(t)} \in T_{\gamma(t)}M$$

is the lifting of the path $\gamma(t)$ via the vector $Y(t)$ to a curve in TM where now

$$G(x^1, \ldots, x^n, Z^1, \ldots, Z^n) := (Z^1, \ldots, Z^n, -Z^pZ^q\Gamma^1_{pq}, \ldots, -Z^pZ^q\Gamma^n_{pq}(x))$$

is a smooth vector field on TM. A solution curve $\tilde{\gamma}(t)$ of (1"),(2") yields a pair $\gamma(t), Y(t)$ solving (1'),(2') and hence a geodesic $\gamma(t)$ (we call it $\gamma_{p,X}(t)$) solving (1),(2). This proves Short Term Existence Theorem for geodesics (as it was stated).

Local flow of G

By ODE theory:

Proposition 9.3 *Fix $p \in M$. Then there exists a open set $U \subseteq M$ with $p \in U, \varepsilon > 0, \delta > 0$ and $W \subseteq TM$ open of the form*

$$W := \{(x, Z) | x \in U, |Z| < \varepsilon\}$$

and a smooth map

$$\phi : W \times [-\delta, \delta] \to TM$$
$$(x, Z) \in W \ t \in [\delta, \delta]$$

that is the flow for (1"),(2"), i.e.

$$\begin{aligned}\phi(x, Z, 0) &= (x, Z) \\ \frac{\partial \phi}{\partial t}(x, Z, t) &= G(\phi(x, Z, t)) \\ \phi(p, X, t) = (\gamma_{p,X}(t), Y_{p,X}(t))\end{aligned}$$

Smoothness of exp and existence in a neighborhood of 0 in T_pM

$$\gamma_{x,Z}(t) = \pi(\phi(x, Z, t)), \pi : TM \to M$$

We have

$$\begin{aligned}\exp_x(Z) &= \gamma_{x,Z}(1) \\ &= \gamma_{x,Z/\delta}(\delta) \\ &= \pi(\phi(x, Z/\delta, \delta)) \quad \left|\frac{Z}{\delta}\right| < \varepsilon\end{aligned}$$

Thus $\exp_x(Z)$ is defined for $x \in U$, $|Z| < \varepsilon\delta$ and is smooth in both variables. Set $B_r^{T_pM}(0) := \{X \in T_pM, |X| < r\}$

Lemma 9.4 $\exp_p : B_r^{T_pM}(0) \to M$ *is defined and smooth for sufficiently small* $r > 0$.

Theorem 9.5 *For each* $p \in M$ $\exists \varepsilon > 0$ *such that* $\exp_p : B_\varepsilon^{T_pM}(0) \to M$ *is a diffeomorphism onto its (open) image. In fact,*

$$(d\exp_p)_0 : \underbrace{T_0 T_p M}_{T_pM} \to T_pM$$

is the identity.

Proof of Theorem By Inverse Function Theorem, it suffices to prove the latter statement. The path

$$t \mapsto tX \text{ in } T_pM$$

goes to the path

$$t \mapsto \gamma(t) := \exp_p(tX) \text{ in } M$$

which is a geodesic in M with $\gamma(0) = p, \dot\gamma(0) = X$.

Differentiate:

$$\begin{aligned} X &= \dot\gamma(0) \\ &= \frac{d}{dt}\exp_p(tX) \\ &= (d\exp_p)_0\left(\left.\frac{dt}{dt}\right|_0 (tX)\right) \\ &= (d\exp_p)_0(X) \end{aligned}$$

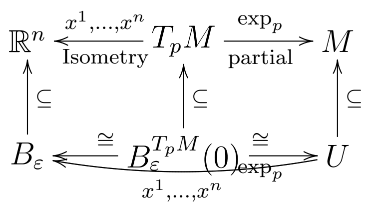
□

Exponential Coordinates

- geodesic normal coordinates
- geodesic polar coordinates

Geodesic Normal Coordinates

Let $x^1, \ldots, x^n$ be orthonormal coordinates on the inner product space $(T_pM, g(p))$. Transfer these coordinates to M via $\exp_p^{-1}$ to obtain *geodesic normal coordinates* near p:



$$g(X,Y) = g_{ij}(x)X^iY^j$$
$$\delta(X,Y) = \delta_{ij}X^iY^j = X^iY^i$$

Compare

$$g = (g_{ij}(x)), x \in U$$

(expressed in exponential normal coordinates) to $\delta = (\delta_{ij})$ (the back ground flat metric coming from $x^1, \ldots, x^n$.)

Theorem 9.6 *In geodesic normal coordinates at p,*

$$g_{ij}(0) = \delta_{ij}, \frac{\partial g_{ij}}{\partial x^k}(0) = 0, \Gamma^k_{ij}(0) = 0.$$

So $g_{ij}(x) = \delta_{ij} + \mathcal{O}(|x|^2)$ [21] *for* $x \in U$ *near* p. *"Metric looks Euclidean up to 1st order".*

[21] $|x| = |x|_\delta = \sqrt{x^ix^i}$, $\mathcal{O}$ is some $\varepsilon_{ij}(x)$ such that $|\varepsilon_{ij}(x)| \leq c|x|^2$

Consequence

A Riemannian metric has no first order invariants to distinguish it from flat space (Euclidean space).

Proof

i. $g_{ij}(p) = \langle \left(\frac{\partial}{\partial x^i}\right)_p, \left(\frac{\partial}{\partial x^j}\right)_p \rangle = \delta_{ij}$ since we chose orthonormal coordinates $x^1, \ldots, x^n$ on T_pM.

ii. Fix $X = X^i \left(\frac{\partial}{\partial x^i}\right)_p \in T_pM$. Consider the geodesic

$$\gamma(t) = \exp_p(tX)$$

with $\dot{\gamma}(0) = X$. In geodesic normal coordinates, $\gamma(t)$ is given by

$$\begin{aligned} \gamma(t) &= (tX^1, \ldots, tX^n) \\ \dot{\gamma}(t) &= (X^1, \ldots, X^n) \qquad \left(= X^i \left(\tfrac{\partial}{\partial x^i}\right)_{\gamma(t)} \in T_{\gamma(t)}M\right) \end{aligned}$$

i.e. $\dot{\gamma}(t)$ agrees along γ with the constant coefficent vector field

$$\tilde{X}(q) := X^i \left(\frac{\partial}{\partial x^i}\right)_q, q \in U$$

$$\tilde{X}(\gamma(t)) = \dot{\gamma}(t).$$

Since γ is a geodesic,

$$0 = \ddot{\gamma}(t) = \mathcal{D}_{\dot{\gamma}}\dot{\gamma}(t) = \left(\mathcal{D}_{\tilde{X}}\tilde{X}\right)(\gamma(t))$$

At $t = 0$:

$$0 = \mathcal{D}_{\tilde{X}}\tilde{X}(0)^k = \underbrace{X^i \frac{\partial X^k}{\partial x^i}}_{=0} + X^i X^j \Gamma^k_{ij}(0)$$

i.e.

$$\Gamma^k_{ij}(0) X^i X^j = 0, \ \forall k.$$

Since this holds $\forall X$ and Γ^k_{ij} is symmetric, polarization yields

$$\Gamma^k_{ij}(0) = 0 \ \forall i, j, k.$$

iii. Compute on U:

$$\begin{aligned}\frac{\partial g_{jk}}{\partial x^i} &= \frac{\partial}{\partial x^i}\langle \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\rangle \\ &= \langle \mathcal{D}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\rangle + \langle \frac{\partial}{\partial x^j}, \mathcal{D}_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}\rangle \\ &= \langle \Gamma^\ell_{ij} \frac{\partial}{\partial x^\ell}, \frac{\partial}{\partial x^k}\rangle + \langle \frac{\partial}{\partial x^j}, \Gamma^\ell_{ik} \frac{\partial}{\partial x^\ell}\rangle \\ &= 0 \quad \text{at } x = 0 \text{ by}(ii)\end{aligned}$$

□

Remark on polarization Let $A(X, Y)$ be symmetric, then

$$A(X, Y) = \frac{1}{2}\left(A(X+Y, X+Y) - A(X, X) - A(Y, Y)\right)$$

Exercise (Lee)
Show: if two connections on TM (not necessarily torsion free!) have the same *symmetric part*, then they have the same geodesics.

Corollary 9.7 *Any vector X in T_pM can be extended to $\tilde{X} \in C^\infty(T_pU), p \in U$ such that $\tilde{X}$ is parallel at p, i.e.*

$$\mathcal{D}_Y \tilde{X}(p) = 0 \ \forall Y.$$

Geodesic Polar Coordinates

Place polar coordinates on T_pM and transfer them to $U \subseteq M$ via $\exp_p^{-1}$. Let $S^{n-1} :=$ unit sphere in T_pM (identified with standard unit sphere in $\mathbb{R}^n$). Define

$$\begin{aligned}[0, \infty) \times S^{n-1} &\rightarrow T_pM \\ (r, \omega) &\mapsto r\omega\end{aligned}$$

Obtain coordinates $r, \omega^1, \ldots, \omega^{n-1}$ and coordinate vector fields $\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega^1}, \ldots, \frac{\partial}{\partial w^{n-1}}$ on $U \setminus \{p\} \subseteq M$. Write $S(r) = \{r\} \times S^{n-1}$.

Lemma 9.8 *In $U \setminus \{p\}$, with respect to g:*

i. $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r}\rangle = 1$

ii. $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \omega^a}\rangle = 0,\ a = 1, \ldots, n-1$

Radial geodesics $t \mapsto t\omega$ are othognoal to coordinate spheres $S(r)$.

iii. $\langle \frac{\partial}{\partial \omega^a}, \frac{\partial}{\partial \omega^b} \rangle = \mathcal{O}(r^2)$

Proof

i. Fix $\omega \in S^{n-1}$. Then $\gamma(t) := \exp_p(t\omega), t \in \mathbb{R}$ is a geodesic with coordinate expression
$$t \mapsto (t, \omega^1, \ldots, \omega^{n-1}) \ (t \neq 0)$$
Thus
$$\dot{\gamma}(t) = (1, 0, \ldots, 0) = \left(\frac{\partial}{\partial r}\right)_{\gamma(t)} \quad (t \neq 0)$$
so
$$\begin{aligned} \left|\frac{\partial}{\partial r}\right|_{\gamma(t)} &\overset{t\neq 0}{=} |\dot{\gamma}|_{\gamma(t)} \\ &= \text{const} \end{aligned}$$
since γ is a geodesic. What is this constant?
Remember: $\left|\frac{\partial}{\partial r}\right|_\delta = 1$ (pre-DG fact) so
$$\begin{aligned} \left|\frac{\partial}{\partial r}\right|_g &= \left|\frac{\partial}{\partial r}\right|_\delta (1 + \mathcal{O}(|x|^2)) \\ &= 1 + \mathcal{O}(|x|^2) \end{aligned}$$
($r = |x|$, $|x|$ means $|x|_\delta$) so the constant is 1.

ii. Fix $a \in \{1, \ldots, n-1\}$ To show: $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \omega^a} \rangle = 0$ on $U \setminus \{p\}$.
Observe:
$$\mathcal{D}_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \omega^a} - \mathcal{D}_{\frac{\partial}{\partial \omega^a}} \frac{\partial}{\partial r} = [\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega^a}] = 0 \text{ on } U \setminus \{p\}$$
$r(\gamma(t)) = t, \frac{\partial}{\partial r} = \frac{d}{dt}$. Now consider $\frac{\partial}{\partial r}, \frac{\partial}{\partial \omega^a}$ as vector fields along $\gamma(t) = \exp_p(t\omega), (\dot{\gamma} = \frac{\partial}{\partial r})$. Compute
$$\begin{aligned} \frac{d}{dt} \langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \omega^a} \rangle_{\gamma(t)} &= \langle \overbrace{\mathcal{D}_{\frac{\partial}{\partial r}} \frac{\partial}{\partial r}}^{=\ddot{\gamma}=0}, \frac{\partial}{\partial \omega^a} \rangle + \langle \frac{\partial}{\partial r}, \mathcal{D}_{\frac{\partial}{\partial r}} \frac{\partial}{\partial \omega^a} \rangle \\ &= 0 + \langle \frac{\partial}{\partial r}, \mathcal{D}_{\frac{\partial}{\partial \omega^a}} \frac{\partial}{\partial r} \rangle \\ &= \frac{1}{2} \frac{\partial}{\partial \omega^a} \cdot \underbrace{\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \rangle}_{\equiv 1} = 0 \end{aligned}$$

so $\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \omega^a} \rangle = \text{const}$ along γ. What is this constant?

$$\begin{aligned} |\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \omega^a} \rangle| &\leq \left| \frac{\partial}{\partial r} \right|_g \left| \frac{\partial}{\partial \omega^a} \right|_g \quad \text{Cauchy-Schwarz} \\ &= 1 \cdot \mathcal{O}(r) \end{aligned}$$

so the constant is zero.

iii. Note $\langle \frac{\partial}{\partial \omega^a}, \frac{\partial}{\partial \omega^b} \rangle_\delta = r^2 h^\circ_{ab}(w)$ (standard metric on S^{n-1}). Since $g_{ij} = \delta_{ij} + \varepsilon_{ij}, \varepsilon_{ij} = \mathcal{O}(r^2)$, where $|\varepsilon_{ij}(r, \omega)| \leq Cr^2$

$$\langle \frac{\partial}{\partial \omega^a}, \frac{\partial}{\partial \omega^b} \rangle_g = r^2 h^\circ_{ab}(\omega) + \mathcal{O}(r^2) = \mathcal{O}(r^2)$$

□

Corollary 9.9 (Gauss's Lemma) *In geodesic polar coordinates, g has the form*

$$g = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & r^2 h_{ij}(r, \omega) & \\ 0 & & & \end{pmatrix} \begin{matrix} r \\ \omega^1 \\ \vdots \\ \omega^{n-1} \end{matrix}$$

where for each $r > 0, h_{ij}(r, \cdot)$ is a metric on S^{n-1} with

$$h_{ij}(r, \omega) = h^\circ_{ij}(\omega) + \mathcal{O}(r^2)$$

as $r \to 0$.

Proof A slight refinement of the above.

□

9.2 Length-minimizing curves

$$L(\gamma) := \int_a^b |\dot{\gamma}(t)|_g dt,$$
$$\gamma : [a, b] \to M.$$

The curve γ is *length-minimizing* if

$$L(\gamma) \leq L(\beta)$$

for any smooth curve β with the same endpoints (resp. *strictly length-minimizing* if equality implies $\beta = \gamma$).

Theorem 9.10 (Local Length-minimizing Property) *Let γ be geodesic Then for each $a \in dom(\gamma)$ and each b sufficiently close to a ($b > a$) $\gamma|[a,b]$ is length-minimizing.*

Example $\alpha = \gamma|[a,b]$. α is length-minimizing iff $L(\alpha) \leq \pi$ (strictly length-minimizing iff $L(\alpha) < \pi$)

Proof Without loss of generality $a = 0$. Set $p = \gamma(0)$. Select $\varepsilon > 0$ such that $\exp_p : B_\varepsilon^{T_pM}(0) \overset{\cong}{\to} U \subseteq M$ is a diffeomorphism. Fix $b < \varepsilon, q := \gamma(b)$. Use geodesic normal coordinates on U. In these coordinates, $\gamma(t), 0 \leq t \leq b$ is the ray $t \mapsto (tX^1, \ldots tX^n)$ where $X := \dot\gamma(0)$. Let β by any curve connectiong $p = \gamma(0)$ to $q = \gamma(b)$.
$L(\gamma|[0,b]) = b$ To show: $L(\beta) \geq b$. Without loss of generality replace β by the initial segment $\beta|[0,e]$ such that

$$\beta(e) \in S(b), \beta([0,e]) \subseteq \{r(x) \leq b\}$$

Show: $L(\beta|[0,e]) \geq b$. Write

$$\begin{aligned}
\beta(u) &= \left(r(u), \omega^1(u), \ldots, \omega^{n-1}(u)\right), 0 \leq u \leq e \\
\dot\beta(u) &= \left(\frac{dr}{du}, \frac{d\omega^1}{du}, \ldots, \frac{d\omega^{n-1}}{du}\right) \\
&= \underbrace{\frac{dr}{du}\frac{\partial}{\partial r}}_{\text{radial part}} + \underbrace{\sum_{a=1}^{n-1} \frac{d\omega^a}{du}\frac{\partial}{\partial \omega^a}}_{\text{tangental part}} \\
&= \dot\beta(u)^R + \dot\beta(u)^T
\end{aligned}$$

so

$$\begin{aligned}
|\dot\beta(u)|^2 &= |\dot\beta(u)^R|^2 + |\dot\beta(u)^T|^2 \\
|\dot\beta(u)| &\geq |\frac{dr}{du}||\frac{\partial}{\partial r}| = |\frac{dr}{du}|
\end{aligned}$$

so

$$\begin{aligned}
L(\beta|[0,e]) &= \int_0^e |\dot\beta(u)|\, du \\
&\geq \int_0^e |\frac{dr}{du}|\, du \\
&\geq r(e) - r(0) \\
&= b - 0 = b
\end{aligned}$$

□

Furthermore: equality occurs iff $\dot{\beta}$ is a nonnegative multiple of $\frac{\partial}{\partial r}$ for all $u \in [0, e]$. But then, $\beta = \gamma[0, b]$! γ is a strict minimizer,$b < \varepsilon$!
Recall $d(p, q) := \inf\{L(\beta)|\beta \text{joins } p \text{ to } q\}$

Definition If $\exp_p : B_\varepsilon^{T_pM}(0) \xrightarrow{\cong} U \subseteq M$ is a diffeomorphism, we call U a *normal neighborhood of* p.

Corollary 9.11 *$p, q \in M$, $r < \varepsilon$ normal coordinates about p.*

$$\begin{array}{rcll} d(p,q) & = & r(q) & \textit{if } q \in \exp_p(B_\varepsilon^{T_pM}(0)) \\ d(q,p) & \geq & \varepsilon & \textit{if } q \notin \exp_p(B_\varepsilon^{T_pM}(0)) \end{array}$$

9.3 Metric Space Structure

(induced by g)
$(M, g) \rightsquigarrow d(q, p)$.

Proposition 9.12 *(M connected) (M, d) is a metric space. (M not connected:* extended metric space*: $d = \infty$ allowed.)*

Proof

- Triangle inequality:$d(x, y) + d(y, z) \geq d(x, z)$
- symmetry: $d(p, q) = d(q, p)$
- positivity: if $p \neq q$ then $d(p, q) > 0$.

□

Proof $p \neq q$, pick ε so $q \notin \exp_p(B_\varepsilon^{T_pM}(0))$ $d(q, p) \geq \varepsilon$.

□

Definition

$$B_\sigma(p)(= B_\sigma^g(p) = B_\sigma^M(p)) := \{q \in M | d(p, q) < \sigma\}$$

geodesic ball of radius σ about p.

Example (need not be a topological ball) By the Corollary(9.11):

$$B_\varepsilon(p) = \exp_p(B_\varepsilon^{T_pM}(0))$$

(provided $\exp_p |B_\varepsilon^{T_pM}(0)$ is a diffeomorphism onto it's image.)

This implies

Proposition 9.13 *The metric space topology generated by $d(\cdot,\cdot)$ coincides with the topology induced by the differntial structure.*

Proof Both topologes are generated (by taking arbitrary unions) by small balls $B_\sigma(p), \sigma < \varepsilon(p)$.

□

Theorem 9.14 (Geodesically Convex Balls) *For $p \in M$, there is $\sigma = \sigma(p) > 0$ such that every pair of points $p_1, p_2 \in B_\sigma(p)$ can be joined by a (unique) minimizing geodesic γ, and γ lies in $B_\sigma(p)$.*

Completeness: Hopf-Rinow Theorem

Questions:

- When can geodesics be extended indefinitely
- When can $p, q \in M$ be joined by a minimizing geodesic?

Theorem 9.15 (Hopf-Rinow) *(M, g) The following are equivalent:*

i. *(M, d) is* metrically complete *(cauchy sequences converge).*

ii. *(M, g) is* geodesically complete *(each geodesic can be extended indefinitely)*

We call M complete.

Example Any compact manifold is complete.

Example $\mathbb{R}^2 \setminus \{0\}$. Metric completion: $\mathbb{R}^2$.
$\widetilde{\mathbb{R}^2 \setminus \{0\}}$ metric completion $\widetilde{\mathbb{R}^2 \setminus \{0\}} \cup \{z\}$

Corollary 9.16 (of Proof) *M connected, complete $\Rightarrow$ every pair p, q can be joined by a minimum geodesic. $\Leftrightarrow$ $\exp_p$ is surjective for all p, i.e. there are no places you can't see from p.*

Example Hyperbolic space is complete.

Proposition 9.17 *If a curve $\gamma \subseteq M^2$ is the fixed-point of a nontrivial isometry, then that curve is a geodesic.*

10 Testing for Flatness

(Lee chap 7) (Motivation for Riemannian curvature tensor.)
How can we tell when 2 Riemannian manifolds are locally isometric? Answer: Invariants.

10.1 Special case

How can we tell when a Riemannian manifold is flat (= locally isometric to Euclidean space)?

Observation

If M is flat, then near each point there is a frame $e_1(x), \ldots, e_n(x)$ consisting of parallel vector fields.

$$
\begin{array}{ccc}
(\mathbb{R}^n, \delta) \subseteq V & \overset{\text{isom. } \phi}{\longleftarrow} & U \subseteq (M^n, g) \\
\dfrac{\partial}{\partial x^i} & \mapsto & \phi^*(\dfrac{\partial}{\partial x^i})
\end{array}
$$

$$\phi^*(\mathcal{D}^{\delta}_X Y) = \mathcal{D}^{\phi^*(\delta)}_{\phi^*(X)} \phi^*(Y)$$

Theorem 10.1 *No neighborhood of a point in S^2 possesses a parallel vector field.* Thus*: No neighborhood af any point in S^2 is isometric to an open set in $\mathbb{R}^2$.*

Lemma 10.2 *The holonomy about a circle of latitude $\gamma = \partial B^{S^2}_\theta(N)$ is a nontrivial rotation*

$$H\gamma : T_pS^2 \to T_pS^2$$

Proof sketch (Do Carmo) Let C be the cone tangent to S^2 along γ. Since S^2 and C have the same tangent planes along γ, we have for any vector field $X(t) \in T_{\gamma(t)}S^2$ along γ

$$\mathcal{D}^{S^2}_{\dot\gamma} X = \pi^{\perp}\left(\mathcal{D}^{\mathbb{R}^3}_{\dot\gamma} X\right) = \mathcal{D}^{C}_{\dot\gamma} X$$

So the holonomy about γ is the same, whether we regard γ as a curve in S^2 or in C. But C can be cut and rolled out flat and the holonomy computed easily.

□

Exercise Find the holonomy about any simple closed curve in S^2.

$$\begin{array}{ccc} \mathbb{C} & \longrightarrow & E \\ & & \downarrow \\ & & M^2 \end{array}$$

$$(\mathcal{D}_X V)^\alpha = X^i \frac{\partial V^\alpha}{\partial x^i} + i \underbrace{\omega(X)}_{\Delta} V^\alpha$$

$$V \in C^\infty(E) \quad \omega(X) = a(x)X^1(x) + b(x)X^2(x)$$

$$\begin{array}{rcl} H_\gamma : E_p & \to & E_p \cong \mathbb{C} \cong \mathbb{R}^2 \\ \hat{V} & \mapsto & e^{i\int_\Omega \left(a_{x^2}(x^1,x^2) - b_{x^1}(x^1,x^2)\right)dx^1dx^2}\hat{V} \end{array}$$

$$a_{x^2} - b_{x^1} = \text{curl}(a,b)(\equiv \text{rot}(a,b))$$

10.2 Try to construct a parallel vector field (locally)

(M^2, g) given, $p \in M$ fixed. x^1, x^2 local coords near p. Fix $Z \in T_pM$. Extend Z parallel along x^1-axis $t \mapsto (t, 0)$. Then extend vertically along each curve $t \mapsto (x^1, t)$ ($x^1 \in \mathbb{R}$ fixed). Get:

$$\begin{cases} \mathcal{D}_{\frac{\partial}{\partial x^2}} Z = 0 & \text{all } x^1, x^2 \\ \mathcal{D}_{\frac{\partial}{\partial x^1}} Z = 0 & \text{all } x^1, x^2 = 0. \end{cases}$$

If $\mathcal{D}_{\frac{\partial}{\partial x^1}} Z = 0$ for all x^1, x^2 then Z would be parallel:

$$\mathcal{D}_X Z = X^1 \mathcal{D}_{\frac{\partial}{\partial x^1}} Z + X^2 \mathcal{D}_{\frac{\partial}{\partial x^2}} Z$$

Too see what $\mathcal{D}_{\frac{\partial}{\partial x^1}} Z$ is like for $x^2 \neq 0$, consider how it varies along curve $t \mapsto (x^1, t)$. Measured by

$$\mathcal{D}_{\frac{\partial}{\partial x^2}} \mathcal{D}_{\frac{\partial}{\partial x^1}} Z$$

Now if we were so lucky and the operators $\mathcal{D}_{\frac{\partial}{\partial x^2}}, \mathcal{D}_{\frac{\partial}{\partial x^1}}$ *commuted* on Z, then

$$\mathcal{D}_{\frac{\partial}{\partial x^2}} \mathcal{D}_{\frac{\partial}{\partial x^1}} Z = \mathcal{D}_{\frac{\partial}{\partial x^1}} \underbrace{\mathcal{D}_{\frac{\partial}{\partial x^2}} Z}_{0} = 0 \ \forall x^1, x^2$$

Then $\mathcal{D}_{\frac{\partial}{\partial x^1}} Z$ would be *parallel* along $t \mapsto (x^1, t)$. But $\mathcal{D}_{\frac{\partial}{\partial x^1}} Z = 0$ at $(x^1, 0)$. So $\mathcal{D}_{\frac{\partial}{\partial x^1}} Z$ would be $0 \ \forall x^1, x^2$.

So the question of constructing parallel vector fields comes down to: *Do directional derivatives of vector fields commute?*
In $\mathbb{R}^n$, this is true: $\mathcal{D}^\delta = \mathcal{D}^0 =$ coordinate connections.

$$\begin{aligned}\mathcal{D}^0_{\frac{\partial}{\partial x^1}}\mathcal{D}^0_{\frac{\partial}{\partial x^2}}\left(Z^i(x)\frac{\partial}{\partial x^i}\right) &= \mathcal{D}_{\frac{\partial}{\partial x^1}}\left(\frac{\partial Z^i}{\partial x^2}(x)\frac{\partial}{\partial x^i}\right)\\ &= \frac{\partial^2 Z^i}{\partial x^1 \partial x^2}(x)\frac{\partial}{\partial x^i}\\ &= \mathcal{D}^0_{\frac{\partial}{\partial x^2}}\mathcal{D}^0_{\frac{\partial}{\partial x^1}}Z\end{aligned}$$

$$\mathcal{D}_X\mathcal{D}_Y Z \stackrel{?}{=} \mathcal{D}_Y\mathcal{D}_X Z$$

Even in $\mathbb{R}^n$, it's not so simple.

$$\begin{aligned}\mathcal{D}^0_X\mathcal{D}^0_Y Z &= X^i\mathcal{D}^0_{\frac{\partial}{\partial x^i}}\left(Y^j\mathcal{D}^0_{\frac{\partial}{\partial x^j}}Z\right)\\ &= X^iY^j\mathcal{D}^0_{\frac{\partial}{\partial x^i}}\mathcal{D}^0_{\frac{\partial}{\partial x^j}}Z + X^i\frac{\partial Y^j}{\partial x^i}\mathcal{D}^0_{\frac{\partial}{\partial x^j}}Z\end{aligned}$$

Antisymmetrizing, we get

$$\begin{aligned}\mathcal{D}^0_X\mathcal{D}^0_Y Z - \mathcal{D}^0_Y\mathcal{D}^0_X Z &= O + [X,Y]^j\mathcal{D}_{\frac{\partial}{\partial x^j}}Z\\ &= \mathcal{D}^0_{[X,Y]}Z.\end{aligned}$$

According:

Proposition 10.3 *In a flat manifold*

$$\mathcal{D}_X\mathcal{D}_Y Z - \mathcal{D}_Y\mathcal{D}_X Z - \mathcal{D}_{[X,Y]}Z = 0. \qquad (\ddagger)$$

Proof $\mathcal{D}$ and $[\cdot,\cdot]$ are both invariant under isometries.

□

10.3 Riemann Curvature

Definition Let $X, Y, Z, W \in C^\infty(TM)$.

i. The *Riemann curvature operator* of (M, g) is defined as

$$\mathcal{R}(X,Y)Z := -\mathcal{D}_X\mathcal{D}_Y Z + \mathcal{D}_Y\mathcal{D}_X Z + \mathcal{D}_{[X,Y]}Z$$

ii. The *Riemannian curvature tensor* is defined by

$$\mathcal{R}_m(X,Y,Z,W) := \langle\mathcal{R}(X,Y)Z, W\rangle$$

$$\mathcal{R}(\cdot,\cdot)\cdot : C^\infty(TM)\times C^\infty(TM)\times C^\infty(TM)\to C^\infty(TM)$$

$\mathcal{R}_m \equiv 0$ iff M is flat, (iff later).

$\mathcal{R}_m$ measures how far M is from being Euclidean.

10.4 Tensors (over $\mathbb{R}$)

V, W vector spaces with bases $e_1, \ldots e_m$ and $d_1, \ldots, d_n$. $V \otimes W$ vector space $mn = \dim$ basis $e_i \otimes d_j$ $i = 1, \ldots, m, j = 1, \ldots, n$.
$\binom{k}{0}$ tensor over V is a $k-$linear map

$$T : \underbrace{V \times \cdots \times V}_{k} \to \mathbb{R}$$

or equivalently an element of $\underbrace{V^* \otimes \cdots \otimes V^*}_{k}$. Typical element: $T = T_{i_1 \ldots i_m} e^*_{i_1} \otimes \cdots \otimes e^*_{i_m}$, $e^*_1, \ldots, e^*_m$ dual basis (to $e_1, \ldots, e_m$) of $V^*, e^*_i(X) = X^i$ $X_\ell = X^p_\ell e_p$

$$\begin{aligned} T(X_1, \ldots, X_m) &= T_{i_1 \ldots i_m} \left(e^*_{i_1} \otimes \cdots \otimes e^*_{i_m}\right)(X_1, \ldots, X_m) \\ &= T_{i_1 \ldots i_m} e^*_{i_1}(X_1) \cdots e^*_{i_m}(X_m) \\ &= T_{i_1 \ldots i_m} X^{i_1}_1 \cdots X^{i_m}_m. \end{aligned}$$

A $\binom{k}{\ell}$ *tensor over* V is a $k-$linear map

$$\underbrace{V \times \cdots \times V}_{k} \to \underbrace{V \otimes \cdots \otimes V}_{\ell}$$

or equivalently, an element of $\underbrace{V^* \otimes \cdots \otimes V^*}_{k} \otimes \underbrace{V \otimes \cdots \otimes V}_{\ell}$. Given smooth vector bundles $E, F \to M$, we can form smooth vector bundles $E^*, E \otimes F$ over M with fibers

$$(E^*)_p := (E_p)^*, (E \otimes F)_p := E_p \otimes F_p$$

$$T^*M = (TM)^*, T^*_p M = (T_p M)^*.$$

Then a $\binom{k}{\ell}$ *tensor field* T on M is a section

$$T \in C^\infty(\underbrace{T^*M \otimes \cdots \otimes T^*M}_{k} \otimes \underbrace{TM \otimes \cdots \otimes TM}_{\ell})$$

Exercise

i. $\binom{0}{1}$ tensor fields are vector fields

ii. $\binom{1}{0}$ tensor fields are dual vector fields, or 1-forms

iii. g (Riemannian metric) is a $\binom{2}{0}$ tensor field.

$\mathcal{D}_X Y$ vector field in $C^\infty(TM)$

$$\begin{aligned}\mathcal{D}Y &= (\mathcal{D}Y(p) : T_pM \to T_pM) \\ &\in C^\infty(\mathrm{Lin}(TM;TM)) \\ &\in C^\infty(T^*M \otimes TM)\end{aligned}$$

so if Y is a vector field, then $\mathcal{D}Y$ is a $\binom{1}{1}$ tensor field.

$$Z = T(X,Y) := \mathcal{D}^1_X Y - \mathcal{D}^2_X Y \in C^\infty(TM)$$

$T(X,Y)(p)$ depends only on $X(p), Y(p)$ (bilinearly). $T \in C^\infty(T^*M \otimes T^*M \otimes TM)$. So T (the difference between two connections) is a $\binom{2}{1}$ tensor.
$\mathcal{R}(\cdot,\cdot) : C^\infty(TM) \times C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM)$

$$\begin{aligned}\mathcal{R}(X,Y)Z &:= -\mathcal{D}_X\mathcal{D}_Y Z + \mathcal{D}_Y\mathcal{D}_X Z + \mathcal{D}_{[X,Y]}Z \\ \mathcal{R}_m(X,Y,Z,W) &:= \langle \mathcal{R}(X,Y)Z, W\rangle\end{aligned}$$

Proposition 10.4 $(\mathcal{R}(X,Y)Z)(p)$ *depends only on* $X(p), Y(p), Z(p)$ *(and not on their derivatives.)*

TM, E vector bundles over M

Definition A $k-$linear map ($k-$linear over $\mathbb{R}$!)

$$T : C^\infty(TM) \times \cdots \times C^\infty(TM) \to C^\infty(E)$$

is called *tensorial* ($k-$linear over $C^\infty(M)$!)

$$T(f_1X_1, \ldots, f_kX_k) = f_1 \cdots f_k T(X_1, \ldots, X_k)\ \forall f_1, \ldots, f_k \in C^\infty(M)$$

Criterion for being a tensor field

If a $k-$linear map (over $\mathbb{R}$)

$$T : \underbrace{C^\infty(TM) \times \cdots \times C^\infty(TM)}_{k} \to C^\infty(E)$$

is in fact $k-$linear *over* $C^\infty(M)$, i.e.

$$T(f_1X_1, \ldots, f_kX_k) = f_1 \cdots f_k T(X_1, \ldots, X_k)\ \forall f_1, \ldots, f_k \in C^\infty(M)$$

(i.e. T is tensorial), *then* T is given by a tensor field, i.e. $T(X_1, \ldots, X_k)(p)$ depends only on $X_1(p), \ldots, X_k(p)$ and in fact there are $k-$linear maps

$$\tilde{T}(p) : T_pM \times \cdots \times T_pM \to E_p$$

such that

$$T(X_1,\dots,X_n)(p) = (\tilde{T}(p))(X_1(p),\dots,X_k(p))$$

Accordingly, the map

$$\tilde{T} : p \mapsto T(p)$$

is a section $\tilde{T} \in C^\infty(T^*M \otimes \cdots \otimes T^*M \otimes E)$. We drop $\tilde{}$ and identify T with $\tilde{T}$.

Proof Let $\frac{\partial}{\partial x^1},\dots,\frac{\partial}{\partial x^n}$ be a coordinate fram for TM defined over some open $U \ni p$.
Fix a *cutoff function* ϕ for p in U i.e. $\phi \in C^\infty(M)$, $\mathrm{spt}\phi \subset\subset U, \phi \equiv 1$ near p.

$$X_i = X_i^j \frac{\partial}{\partial x^j} \text{ on } U \text{ only!}$$

Compute

$$\begin{aligned}
T(X_1,\dots,X_k)(p) &= \underbrace{\phi^{2k}(p)}_{1} T(X_1,\dots,X_k)(p) \\
&= (\phi^{2k}T(X_1,\dots,X_k))(p) \\
&= T(\phi^2 X_1,\dots,\phi^2 X_k)(p) \\
&= \left((\phi X_1^{j_1})\cdots(\phi X_k^{jk}) T(\phi\frac{\partial}{\partial x^{j_1}},\dots,\phi\frac{\partial}{\partial x^{j_k}}\right)(p) \\
&= X_1^{j_1}(p)\cdots X_k^{j_k}(p) T(\phi\frac{\partial}{\partial x^{j_1}},\dots,\phi\frac{\partial}{\partial x^{j_k}})(p)
\end{aligned}$$

depends only on $X_1(p),\dots,X_k(p)$, and indeed, $k-$linear.

□

Remark

- $\phi\frac{\partial}{\partial x^j} \in C^\infty(TM)$ meaning

$$\phi\frac{\partial}{\partial x^j} = \begin{cases} \phi\frac{\partial}{\partial x^j} & \text{on } U \\ 0 & \text{on } M \setminus \mathrm{spt}\phi \text{ (open)} \end{cases}$$

- $\phi X_i^j \in C^\infty(M)$

$$\begin{gathered}
X, Y, Z, W \in C^\infty(TM) \\
\mathcal{R}(\cdot,\cdot)\cdot : C^\infty(TM) \times C^\infty(TM) \times C^\infty(TM) \to C^\infty(TM) \\
\mathcal{R}(X,Y)Z := -\mathcal{D}_X\mathcal{D}_Y Z + \mathcal{D}_Y\mathcal{D}_X Z + \mathcal{D}_{[X,Y]}Z \\
\mathcal{R}_m(X,Y,Z,W) := \langle \mathcal{R}(X,Y)Z, W \rangle
\end{gathered}$$

Proposition 10.5

$$\begin{aligned}\mathcal{R}(\cdot,\cdot)\cdot &\in C^\infty(T^*M\otimes T^*M\otimes T^*M\otimes TM)\\ \mathcal{R}_m &\in C^\infty(T^*M\otimes T^*M\otimes T^*M\otimes T^*M)\end{aligned}$$

Proof If suffices to check $\mathcal{R}(fX,gY)hZ = fgh\mathcal{R}(X,Y)Z$ for $f,g,h\in C^\infty(M)$ (Tensoriality Criterion).
Do h:

$$\begin{aligned}\mathcal{R}(X,Y)(hZ) &\overset{?}{=} h\mathcal{R}(X,Y)Z\\ \mathcal{D}_X\mathcal{D}_Y(hZ) &= \mathcal{D}_X\left((Yh)Z + h\mathcal{D}_Y Z\right)\\ &= \left(X(Yh)\right)Z + (Yh)\mathcal{D}_X Z + (Xh)\mathcal{D}_Y Z + h\mathcal{D}_X\mathcal{D}_Y Z\\ \mathcal{D}_X\mathcal{D}_Y(hZ) &= \text{similar}\ldots\\ \mathcal{D}_{[X,Y]}(hZ) &= \left([X,Y]h\right)Z + h\mathcal{D}_{[X,Y]}Z\\ \mathcal{R}(X,Y)(hZ) &= -h\mathcal{D}_X\mathcal{D}_Y Z + h\mathcal{D}_Y\mathcal{D}_X Z + h\mathcal{D}_{[X,Y]}Z\\ &\quad - (XYh)Z + (YXh)Z + [X,Y]hZ\\ &= h\mathcal{R}(X,Y)Z\end{aligned}$$

Do f,g: similar but shorter

□

Definition Define components of the curvature tensor in a coordinate neighborhood by

$$\mathcal{R}\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k} = \mathcal{R}^\ell_{ijk}\frac{\partial}{\partial x^\ell}$$

$$\mathcal{R}_{ijkl} := \mathcal{R}_m\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^\ell}\Big) = \Big\langle\mathcal{R}\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^\ell}\Big\rangle$$

Then we have

$$\begin{aligned}\mathcal{R}(X,Y)Z &= X^iY^jZ^k\mathcal{R}^\ell_{ijk}\frac{\partial}{\partial x^\ell}\\ \mathcal{R}_m(X,Y,Z,W) &= X^iY^jZ^kW^\ell\mathcal{R}_{ijk\ell}\end{aligned}$$

Note $\mathcal{R}_{ijkl} = g_{pl}\mathcal{R}^p_{ijk}$. $\mathcal{R}$ given by at most n^4 functions.

Invariance under isometries $\phi:(M,g)\to(N,h)$ isometry

$$\mathcal{R}^g_m(X,Y,Z,W)(p) = \mathcal{R}^h_m(\phi_*X,\phi_*Y,\phi_*Z,\phi_*W)(\phi(p))$$

Diffeomorphism invariance

$$\begin{aligned}\phi^*(f) &= f\circ\phi\\ \phi_*(f) &= f\circ\phi^{-1}\end{aligned}$$

$$\phi_*(\mathcal{R}^g_m(X,Y,Z,W)) = \mathcal{R}^{\phi_*(g)}_m(\phi_*X,\phi_*Y,\phi_*Z,\phi_*W)$$

C^∞ functions on $\mathbb{R}$ with compact support

$$f(x) := \begin{cases} e^{-\frac{1}{x}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

f is C^∞

Claim $f^{(k)}(\eta) \to 0$ as $\eta \to \infty$ $\forall k$

$$\begin{aligned} f^{(1)} &= \frac{1}{x^2}e^{-\frac{1}{x}} \quad f^{(k)} = a_k(x)e^{-\frac{1}{x}} \\ f^{(2)} &= (-\frac{2}{x^3} + \frac{1}{x^4})e^{-\frac{1}{x}} \quad |a_k(x)| \leq x^{-2k}(0 \leq x \leq 1) \end{aligned}$$

Proposition 10.6

- $\mathcal{R}^\ell_{ijk} = -\frac{\partial}{\partial x^i}\Gamma^\ell_{jk} + \frac{\partial}{\partial x^j}\Gamma^\ell_{ik} - \Gamma^\ell_{ip}\Gamma^p_{jk} + \Gamma^\ell_{jp}\Gamma^p_{ik}$
- $\mathcal{R}_{ijkl} = g_{\ell m}\mathcal{R}^m_{ijk}$

Proof

i.

$$\begin{aligned} \mathcal{R}^\ell_{ijk}\frac{\partial}{\partial x^\ell} =& \mathcal{R}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^k} \\ =& -\mathcal{D}_{\frac{\partial}{\partial x^i}}\mathcal{D}_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k} + \mathcal{D}_{\frac{\partial}{\partial x^j}}\mathcal{D}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k} \\ & + \mathcal{D}_{[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}]}\frac{\partial}{\partial x^k} \\ =& -\mathcal{D}_{\frac{\partial}{\partial x^i}}(\Gamma^\ell_{jk}\frac{\partial}{\partial x^\ell}) + \mathcal{D}_{\frac{\partial}{\partial x^j}}(\Gamma^\ell_{ik}\frac{\partial}{\partial x^\ell}) \\ =& (-\frac{\partial}{\partial x^i}\Gamma^\ell_{jk})\frac{\partial}{\partial x^\ell} - \Gamma^\ell_{jk}\mathcal{D}_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^\ell} + (\frac{\partial}{\partial x^j}\Gamma^\ell_{ik})\frac{\partial}{\partial x^\ell} + \Gamma^\ell_{ik}\mathcal{D}_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^\ell} \\ =& -\frac{\partial}{\partial x^i}\Gamma^\ell_{jk}\frac{\partial}{\partial x^\ell} - \Gamma^p_{jk}\Gamma^\ell_{ip}\frac{\partial}{\partial x^\ell} + \frac{\partial}{\partial x^j}\Gamma^\ell_{ik}\frac{\partial}{\partial x^\ell} + \Gamma^p_{ik}\Gamma^\ell_{jp}\frac{\partial}{\partial x^\ell} \end{aligned}$$

□

The proposition shows:

$$g_{ij} \xrightarrow{\text{deriv}} \mathcal{D} \xrightarrow{\text{deriv}} \mathcal{R}_m$$

$\mathcal{R}_m$ = combinations of various 0th, 1st and 2nd derivatives of components of the metric tensor $g_{ij}(x)$.

Exercise Find a formula for $\mathcal{R}_{ijk\ell}$ in terms of $g_{ij}, \partial g_{ij}, \partial^2 g_{ij}$ that shows: $\mathcal{R}_{ijk\ell}$ is

- linear in $\frac{\partial^2 g_{ij}}{\partial x^k \partial x^\ell}$
- quadratic in $\frac{\partial g_{ij}}{\partial x^k}$
- nonlinear in g_{ij}.

(recall: same pattern in ODE for geodesics)

10.4.1 Flat Manifolds

(Lee Chap 7.)

Theorem 10.7 (Riemann) *$\mathcal{R}_m \equiv 0$ iff M is* locally isometric to Euclidean space.

Proof ($\Leftarrow$) done
($\Rightarrow$) Suppose $\mathcal{R}_m \equiv 0$ Fix $p \in M$. 4 steps:

i. Build a set of *parallel*, orthonormal ($\mathcal{R}_m \equiv 0$) vector fields $Y_1, \ldots, Y_n$ near p.

ii. Then $[Y_i, Y_j] = 0 \ \forall i, j$.

iii. Then M has a coordinate system $y^1, \ldots, y^n$ near p with $Y^i = \frac{\partial}{\partial y^i}$.

iv. A coordinate system whose coordinate vector fields are orthonormal is the same as an isometry into $\mathbb{R}^n$.

ii. $\mathcal{D}_{Y_i} Y_j = 0 \ \forall i, j$ by i. so $[Y_i, Y_j] = \mathcal{D}_{Y_i} Y_j - \mathcal{D}_{Y_j} Y_i = 0$

iii. If

(a) $Y_1, \ldots, Y_n$ commute

(b) $Y_1, \ldots, Y_n$ linearly independant at p

$\Rightarrow$ there exists a coordinate system. $\phi = (y^1, \ldots, y^n) : U \subseteq M \xrightarrow{\cong} V \subseteq \mathbb{R}^n$ near p such that

$$\underbrace{Y_i}_{\in U \subseteq M} = \phi^*(\underbrace{\frac{\partial}{\partial y^i}}_{\in \mathbb{R}^n})$$

iv. Then $\langle Y_i, Y_j \rangle_g \overset{(1)}{=} \delta_{ij} = \langle \frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j} \rangle_\delta$ so ϕ is an isometry.

□

Follows from:

Subclaim Any $\hat{Y} \in T_pM$ can be extended to parallel vector field near p. Why does it follow? Fix p. $\hat{Y}_1, \dots, \hat{Y}_n \in T_pM$ orthonormal basis. Use subclaim to extend to $Y_1, \dots, Y_n$ parallel defined near p. But $X \cdot \langle Y_i, Y_j \rangle = \langle \mathcal{D}_X Y_i, Y_j \rangle + \langle Y_i, \mathcal{D}_X Y_j \rangle = 0$ so $\langle Y_i, Y_j \rangle = \delta_{ij}$ is constant near p.

Proof of subclaim Let $x^1, \dots, x^n$ be any coordinate system near p.

$$p = 0,\ U = \{x | -\varepsilon < x_i < \varepsilon\}$$

Fix $\hat{Y} \in T_pM$

$$M_k := \left\{(x^1, \dots, x^k, 0 \dots, 0) | -\varepsilon < x_1, \dots, x_k < \varepsilon\right\} \cong \mathbb{R}^k$$

$$\{0\} = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = U$$

Extend $\hat{Y}$ *from* M_0 *to* M_1 by parallel transport along $\gamma : t \mapsto (t, 0, \dots, 0) \in M_1$. Get:

$$\begin{cases} Y : M_1 & \to & TM_1 \\ \mathcal{D}_{\frac{\partial}{\partial x^1}} Y & = & 0 \text{ on } M_1 \end{cases}$$

Extend from M_1 *to* M_2

$$x = (x^1, 0, \dots, 0) \in M_1$$

$$\gamma_x : t \mapsto (x^1, t, 0, \dots, 0) \in M_2$$

Extend Y along γ_x by parallel translation. Get:

$$\left(\begin{cases} Y : M_2 & \to & TM \\ \mathcal{D}_{\frac{\partial}{\partial x^2}} Y & = & 0 \text{ on } M_2 \\ \mathcal{D}_{\frac{\partial}{\partial x^1}} Y & = & 0 \text{ on } M_1 \end{cases}\right)$$

$Y(x_1, x_2, 0, \dots, 0)$ is smooth in x^1, x^2 by smooth dependence of solutions of ODEs on intial conditions (and using the fact that $(x_1, 0, \dots, 0)$ is smooth). Want: $\mathcal{D}_{\frac{\partial}{\partial x^1}} Y = 0$ on M_2. By defintion of curvature

$$\begin{aligned} \mathcal{D}_{\frac{\partial}{\partial x^2}} \mathcal{D}_{\frac{\partial}{\partial x^1}} Y =& \mathcal{D}_{\frac{\partial}{\partial x^1}} \mathcal{D}_{\frac{\partial}{\partial x^2}} Y + \mathcal{D}_{[\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}]} Y - \mathcal{R}(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}) Y \\ =& \mathcal{D}_{\frac{\partial}{\partial x^1}}, \underbrace{\mathcal{D}_{\frac{\partial}{\partial x^2}} Y}_{=0} \\ =& 0 \text{ on } M_2 \end{aligned}$$

So $\mathcal{D}_{\frac{\partial}{\partial x^1}} Y$ is parallel along γ_x. But $\mathcal{D}_{\frac{\partial}{\partial x^1}} Y = 0$ at $\gamma_x(0) = (x^1, 0, \ldots, 0)$ so $\mathcal{D}_{\frac{\partial}{\partial x^1}} Y = 0$ on γ_x i.e. on M_2.
Proceed by induction.
Extend Y from M_k to M_{k+1} Given:

$$(H_k) \begin{cases} Y : M_k & \to & TM \\ \mathcal{D}_{\frac{\partial}{\partial x^1}} Y & = \cdots = & \mathcal{D}_{\frac{\partial}{\partial x^k}} Y = 0 \text{ on } M_k \end{cases}$$

Want:

$$(H_{k+1}) \begin{cases} Y : M_{k+1} & \to & TM \\ \mathcal{D}_{\frac{\partial}{\partial x^1}} Y & = \cdots = & \mathcal{D}_{\frac{\partial}{\partial x^{k+1}}} = 0 \text{ on } M_{k+1} \end{cases}$$

Using parallel transport along curves

$$\gamma_x : t \mapsto \quad (x^1, \ldots, x^k, t, 0, \ldots, 0) \quad \in M_{k+1}$$
$$(x = (x^1, \ldots, x^k, 0, \ldots, 0) \in M_k$$

get

$$Y : M_{k+1} \to TM$$
$$\mathcal{D}_{\frac{\partial}{\partial x^{k+1}}} Y = 0 \text{ on } M_{k+1}$$

Using $\mathcal{R}_m \equiv 0$ as before, we get

$$\mathcal{D}_{\frac{\partial}{\partial x^{k+1}}} \mathcal{D}_{\frac{\partial}{\partial x^i}} Y = \mathcal{D}_{\frac{\partial}{\partial x^i}} \underbrace{\mathcal{D}_{\frac{\partial}{\partial x^{k+1}}} Y}_{=0} = 0$$

on M_{k+1}, so as (before)

$$\mathcal{D}_{\frac{\partial}{\partial x^i}} Y = 0 \text{ on } M_{k+1} \; \forall i$$

□

10.5 Symmetries of Curvature

i.

$$\mathcal{R}_m(X, Y, Z, W) \overset{(a)}{=} -\mathcal{R}_m(Y, X, Z, W)$$
$$\overset{(b)}{=} -\mathcal{R}_m(X, Y, W, Z)$$

ii. $\mathcal{R}_m(X, Y, Z, W) = \mathcal{R}_m(Z, W, X, Y)$

iii. $0 = \mathcal{R}_m(X, Y, Z, W) + \mathcal{R}_m(Y, Z, X, W) + \mathcal{R}_m(Z, X, Y, W)$ (Bianchi I)

Proof

i. (a) $\mathcal{R}(X, Y)Z = -\mathcal{D}_X\mathcal{D}_Y Z + \mathcal{D}_Y\mathcal{D}_X Z + \mathcal{D}_{[X,Y]}Z$

(b) Differentiate $\langle Z, W\rangle$ twice:

$$\begin{aligned} X \cdot Y \cdot \langle Z, W\rangle &= X \cdot (\langle \mathcal{D}_Y Z, W\rangle + \langle Z, \mathcal{D}_Y W\rangle) \\ &= \langle \mathcal{D}_X\mathcal{D}_Y Z, W\rangle + \langle \mathcal{D}_Y Z, \mathcal{D}_X W\rangle + \langle \mathcal{D}_X Z, \mathcal{D}_Y W\rangle \\ &\quad + \langle Z, \mathcal{D}_X\mathcal{D}_Y W\rangle \end{aligned}$$

Antisymmetrize in X, Y:

$$\begin{aligned} [X, Y] \cdot \langle Z, W\rangle &= \langle \mathcal{D}_X\mathcal{D}_Y Z - \mathcal{D}_Y\mathcal{D}_X Z, W\rangle \\ &\quad + \langle Z, \mathcal{D}_X\mathcal{D}_Y W - \mathcal{D}_Y\mathcal{D}_X W\rangle \end{aligned}$$

$$[X, Y] \cdot \langle Z, W\rangle = \langle \mathcal{D}_{[X,Y]}Z, W\rangle + \langle Z, \mathcal{D}_{[X,Y]}W\rangle$$

Rearrange:

$$\langle \mathcal{R}(X, Y)Z, W\rangle + \langle Z, \mathcal{R}(X, Y)W\rangle = 0$$

iii. (Bianchi I) $0 = \mathcal{R}_m(X, Y, Z, W) + \mathcal{R}_m(Y, Z, X, W) + \mathcal{R}_m(Z, X, Y, W)$.

$$\begin{aligned} \mathcal{R}(X, Y)Z &= -\mathcal{D}_X\mathcal{D}_Y Z + \mathcal{D}_Y\mathcal{D}_X Z + \mathcal{D}_{[X,Y]}Z \\ \mathcal{R}(Y, Z)X &= -\mathcal{D}_Y\mathcal{D}_Z X + \mathcal{D}_Z\mathcal{D}_Y X + \mathcal{D}_{[Y,Z]}X \\ \mathcal{R}(Z, X)Y &= -\mathcal{D}_Z\mathcal{D}_X Y + \mathcal{D}_X\mathcal{D}_Z Y + \mathcal{D}_{[Z,X]}Y \\ \text{Sum} &= -\mathcal{D}_X[Y, Z] - \mathcal{D}_Y[Z, X] - \mathcal{D}_Z[X, Y] + \mathcal{D}_{[X,Y]}Z + \mathcal{D}_{[Y,Z]}X + \mathcal{D}_{[Z,X]}Y \\ &= -[X, [Y, Z]] - [Y, [Z, X]] - [Z, [X, Y]] = 0 \text{ Jacobi identity} \end{aligned}$$

ii. combine i. and iii. cleverly. Exercise

□

In components:

i. $\mathcal{R}_{ijk\ell} = -\mathcal{R}_{jik\ell} = -\mathcal{R}_{ij\ell k}$

ii. $\mathcal{R}_{ijk\ell} = \mathcal{R}_{k\ell ij}$

iii. $\mathcal{R}_{ijk\ell} + \mathcal{R}_{jki\ell} + \mathcal{R}_{kij\ell} = 0$

Elie Carton called Differential Geometry "the debauch of indices". Gromov: "The Riemannian curvature tensor remains a nasty, mysterios bundle of multilinear algebra."

Exercise What is the dimension of the space of potential curvature tensors at a point?

Example

$n = 1$ $\mathcal{R}_{1111} = -\mathcal{R}_{1111} \Rightarrow \mathcal{R}_{1111} \equiv 0$ no curvature.

$n = 2$ $0 = \mathcal{R}_{11ij} = \mathcal{R}_{22ij} = \mathcal{R}_{ij11} = \mathcal{R}_{ij22}$ $\mathcal{R}_{1212} = -\mathcal{R}_{2112} = -\mathcal{R}_{1221} = \mathcal{R}_{2121}$ The Riemannian curvature tensor of a 2−manifold reduces to a single scalar. What is that scalar?

i. (M^2, g) $\kappa(p) := \mathcal{R}_m(e_1, e_2, e_1, e_2)$, e_1, e_2 orthonormal basis of T_pM.

Exercise Prove $\kappa(p)$ is independent of choice of e_1, e_2.

Theorem 10.8 (Theorema Egregium (Gauss)) *Suppose (M^2, g) is isometrically embedded in $\mathbb{R}^3$. Then*

$$\kappa(p) = k_1 \cdot k_2$$

product of principal curvatures of M^2 inside $\mathbb{R}^3$.

$(M^n, g), p \in M, \sigma \subset T_pM$ 2-plane

Definition *Sectinal curvature* of M at p along σ.

$$\kappa(p, \sigma) := \mathcal{R}_m(e_1, e_2, e_1, e_2)$$

e_1, e_2 orthonormal basis of σ. (Exercise: independence of e_1, e_2)

Fact

$$\begin{aligned} \kappa(p, \sigma) &\equiv 1 \quad \text{on } S^n \\ \kappa(p, \sigma) &\equiv -1 \quad \text{in } \mathbb{H}^n \end{aligned}$$

Theorem 10.9 *If (M, g) has $\kappa(p, \sigma) \geq \frac{1}{r^2} > 0\ \forall p, \sigma$ then M is compact and diam$(M) := \max_{p,q \in M} d(p, q) \leq \pi r$ $\kappa \geq \frac{1}{r^2} > 0 \Rightarrow M$ is compact.*

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