The most important exercises are marked with an asterisk *.

1.1. Let $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth Hamiltonian function. Suppose H is compactly supported and let Φ_t be the associated Hamiltonian flow.

(a) Consider the Hamiltonian vector field

$$X(q,p) = \begin{pmatrix} \frac{\partial H}{\partial p}(q,p) \\ -\frac{\partial H}{\partial q}(q,p) \end{pmatrix}.$$

Show that $\operatorname{div} X = 0$.

(b) Let $Y\colon \mathbb{R}^m \to \mathbb{R}^m$ be any smooth vector field with $\operatorname{div} Y = 0$. Suppose $\Psi_t\colon \mathbb{R}^m \to \mathbb{R}^m$ is its flow, i.e.

$$\forall x \in \mathbb{R}^m\colon \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(x) = Y(\Psi_t(x)), \qquad \Psi_0 = \mathrm{id}\,.$$

Show that

$$\det\left(\mathrm{d}\Psi_t(x)\right) = 1 \qquad \text{for all } x \in \mathbb{R}^m \text{ and } t \in \mathbb{R}.$$

Hint: Jacobi's formula on how to express a derivative of a determinant might be helpful.

(c) Deduce from parts (a) and (b) that Φ_t is volume-preserving, i.e. for each bounded open domain $U \subseteq \mathbb{R}^{2n}$

$$\mathrm{vol}(U) = \mathrm{vol}(\Phi_t(U)).$$

This is the Liouville Theorem.

***1.2.** Consider the standard volume form on $\mathbb{R}^{2n}$:

$$\mathrm{vol} = \mathrm{d}p_1 \wedge \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}p_n \wedge \mathrm{d}q_n.$$

Show that

$$\mathrm{vol} = \frac{\omega_{\mathrm{std}}^{\wedge n}}{n!},$$

where

$$\omega_{\mathrm{std}} = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

Deduce that any symplectomorphism $\Phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is volume-preserving.

***1.3.** Let $\Sigma \subseteq \mathbb{R}^3$ be a surface, i.e. a smooth 2-dimensional submanifold, and let $\nu\colon \Sigma \to \mathbb{R}^3$ be a co-orientation of Σ, i.e. a smooth unit normal vector field. We define the 2-form ω on Σ by

$$\omega_x(v, w) \coloneqq \nu(x) \cdot (v \times w) \qquad \forall x \in \Sigma,\ v, w \in T_x\Sigma,$$

where $\cdot$ denotes the Euclidean inner product and $\times$ denotes the cross product.

(a) Show that ω is a symplectic form.

(b) Give a formula for ω in the case of the 2-sphere:

$$S^2 \coloneqq \{x \in \mathbb{R}^3 \mid |x| = 1\}.$$

***1.4.** Let (M, ω) be a $2n$-dimensional symplectic manifold. Show that $\omega^{\wedge n}$ is a volume form, i.e. that $\omega^{\wedge n}$ is a nowhere vanishing form of top degree.

Note that would imply that M is canonically oriented. The form $\frac{\omega^{\wedge n}}{n!}$ is called the *symplectic volume*.