The most important exercises are marked with an asterisk *.

***2.1.** Let (M, ω) be a closed symplectic manifold of positive dimension. Show that ω is not exact.

***2.2.** Let (M, ω) be a symplectic manifold, $H, K\colon [0,1] \times M \to \mathbb{R}$ be two smooth Hamiltonian functions and $\chi \in \mathrm{Symp}(M, \omega)$.

(a) Show that $\psi_t^H \circ \psi_t^K$ is generated by

$$(H\#K)_t = H_t + K_t \circ (\psi_t^H)^{-1}.$$

(b) Show that $\left(\psi_t^H\right)^{-1}$ is generated by

$$\overline{H}_t = -H_t \circ \psi_t^H.$$

(c) Show that $\chi^{-1}\psi_t^H\chi$ is generated by $H_t \circ \chi$.

(d) Deduce from parts (a), (b) and (c) that $\mathrm{Ham}(M, \omega)$ is a normal subgroup of $\mathrm{Symp}(M, \omega)$.

2.3. Find an example of a symplectic manifold (M, ω) (without boundary) and a smooth function $H\colon [0, +\infty) \times M \to \mathbb{R}$ such that the domain $\mathcal{D}_H$ of the Hamiltonian flow ψ^H is not equal to $[0, +\infty) \times M$. This flow is by definition the flow of the time-dependent Hamiltonian vector field X^{H_t}, which is defined by $-\mathrm{d}H^t = \omega(X^{H_t}, \cdot)$.

Find an example in which ψ_t^H is not surjective for some t.

***2.4.** This problem is intended for students who are not yet familiar with the Lie derivative and Cartan's formula. Let $I \subset \mathbb{R}$ be an open interval, M a closed smooth manifold and $f\colon I \times M \to M$ a smooth function such that $f_s \in \mathrm{Diff}(M)$ for all $s \in I$. Let $k \geq 1$ and $\omega \in \Omega^k(M)$ a differential k-form on M.

The goal of this exercise is to prove Cartan's formula:

$$\frac{\mathrm{d}}{\mathrm{d}s} f_s^*\omega = f_s^*(\iota_{X_s}\mathrm{d}\omega + d\iota_{X_s}\omega),$$

where X_s is the time-dependent vector field on M defined by

$$\frac{\mathrm{d}}{\mathrm{d}s} f_s = X_s \circ f_s.$$

For $s \in I$ consider the linear map

$$T_s\colon \Omega^k(I \times M) \to \Omega^{k-1}(M),$$

defined by

$$(T_s\sigma)_x(w_1, \dots w_{k-1}) = \sigma_{(s,x)}((1,0), (0, w_1), \dots, (0, w_{k-1}))$$

for $x \in M$ and $w_1, \dots, w_{k-1} \in T_xM$. Here we use the decomposition

$$T_{(s,x)}(I \times M) \cong T_s\mathbb{R} \oplus T_xM \cong \mathbb{R} \oplus T_xM.$$

(a) Let $f \in C^\infty(I \times M)$ and $\alpha \in \Omega^{k-1}(M)$. Prove that for $\sigma = f\,\mathrm{d}s \wedge \pi^*\alpha$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s} i_s^*\sigma = (T_s\mathrm{d} + \mathrm{d}T_s)\sigma,$$

where $i_s\colon M \to I \times M$, $i_s(x) = (s, x)$ and $\pi\colon I \times M \to M$, $\pi(s, x) = x$.

(b) Let $f \in C^\infty(I \times M)$ and $\beta \in \Omega^k(M)$. Prove that for $\sigma = f\,\pi^*\beta$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} i_s^*\sigma = (T_s\mathrm{d} + \mathrm{d}T_s)\sigma.$$

(c) Deduce from parts (a) and (b) that

$$\frac{\mathrm{d}}{\mathrm{d}s} i_s^*\sigma = (T_s\mathrm{d} + \mathrm{d}T_s)\sigma$$

for every $\sigma \in \Omega^k(I \times M)$.

(d) Prove Cartan's formula.

(e) Let $X \in \Gamma(TM)$ be a smooth vector field on M and let ψ_t be its flow. The Lie derivative $\mathcal{L}_X\omega \in \Omega^k(M)$ is defined by

$$\mathcal{L}_X\omega = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi_t^*\omega.$$

Show that

$$\mathcal{L}_X\omega = \iota_X\mathrm{d}\omega + \mathrm{d}\iota_X\omega.$$