The most important exercises are marked with an asterisk *.

***3.1.**

(a) Find a symplectic manifold and a symplectomorphism on M that **is not** isotopic to the identity. (In particular, such a symplectomorphism is not a Hamiltonian diffeomorphism.)

(b) Find a symplectic manifold and a symplectomorphism on M that **is** isotopic to the identity through symplectomorphisms, but is not a Hamiltonian diffeomorphism.

Hint: Consider translations on a cylinder.

(c) Does there exist a non-Hamiltonian symplectomorphism on S^2 equipped with the standard symplectic form, that is isotopic to the identity through symplectomorphisms?

3.2. This exercise covers two useful facts from differential geometry and algebraic topology.

(a) Let $\omega_t \in \Omega^k(M)$ be a differential k-form and φ_t a smooth isotopy of diffeomorphisms. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\omega_t = \varphi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{\mathrm{d}}{\mathrm{d}t}\omega_t\right),$$

where X_t is the vector field defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = X_t \circ \varphi_t.$$

(b) Let $d \geq 1$ and $\alpha \in \Omega^d(\mathbb{R}^n)$ be a closed d-form, i.e. $\mathrm{d}\alpha = 0$. Show that α is exact, i.e. there exists $\lambda \in \Omega^{d-1}(\mathbb{R}^n)$ such that $\alpha = \mathrm{d}\lambda$. In other words, $\mathrm{H}^d(\mathbb{R}^n;\mathbb{R}) = 0$.

Hint: Use the retraction $f_t(x) = tx$ and the strategy we used in the lecture to show that "strongly isotopic" implies "isotopic".

***3.3.** In this exercise, we prove Moser stability for volume forms. Let M be a closed smooth manifold of dimension m.

(a) Suppose $\mu_t \in \Omega^m(M)$, $t \in [0,1]$, is a smooth family of volume forms on M such that

(i) μ_t is a volume form for each t,

(ii) $\frac{\mathrm{d}}{\mathrm{d}t}\mu_t$ is exact for all $t \in [0,1]$.

Prove that there exists a smooth isotopy $\varphi_t \colon M \to M$ of diffeomorphisms on M satisfying $\varphi_t^*\mu_t = \mu_0$ for all $t \in [0,1]$.

(b) Let $\mu_0, \mu_1 \in \Omega^m(M)$ be two volume forms on M such that

$$\int_M \mu_0 = \int_M \mu_1.$$

Prove that there exists a diffeomorphism $\varphi \colon M \to M$, isotopic to id, satisfying $\varphi^*\mu_1 = \mu_0$.

3.4. Let (Σ, σ) and (Σ', σ') be two closed connected symplectic surfaces. Suppose Σ has total area 1 and Σ' has total area c. Let $a \in \mathbb{R}\backslash 0$. Endow the product manifold $\Sigma \times \Sigma'$ with the symplectic form $\omega_a = a\sigma \oplus a^{-1}\sigma'$.

(a) Show that (M, ω_a) all have the same volume.

(b) Show that there exist a such that (M, ω_1) and (M, ω_a) are not symplectomorphic.

Hint: The Degree Theorem from Algebraic Topology tells us the following. Let X and Y be compact oriented manifolds of same dimension and let $f \colon X \to Y$ be a smooth map. Then every top degree form Ω satisfies

$$\int_X f^*\Omega = \deg f \int_Y \Omega$$

Since in Exercise 3.4 (a) we saw that all (M, ω_a) have the same volume, try instead using the Degree Theorem to compare volumes of the projections on Σ of ω_a calculated directly and as $\varphi^*\omega_1$ for φ a symplectomorphism.