The most important exercises are marked with an asterisk *.

***4.1.** Let $\mathrm{Sp}(2n)$ be the group of symplectic matrices

$$\mathrm{Sp}(2n) = \left\{ A \in \mathrm{GL}(2n, \mathbb{R}) \mid A^T J_0 A = J_0 \right\},$$

where

$$J_0 = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}.$$

(a) Show that if $\Psi \in \mathrm{Sp}(2n)$ then $\Psi^{-1} \in \mathrm{Sp}(2n)$ and $\Psi^T \in \mathrm{Sp}(2n)$.

(b) Show that if $P \in \mathrm{Sp}(2n)$ is a symmetric, positive definite symplectic matrix, then $P^\alpha \in \mathrm{Sp}(2n)$ for every $\alpha \geq 0$, $\alpha \in \mathbb{R}$.

(c) Show that $\mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{U}(n)$ and that the inclusion $\mathrm{U}(n) \subset \mathrm{Sp}(2n)$ is a homotopy equivalence.

Hint: Consider the homotopy $f_t(\Psi) = \Psi \left(\Psi^T \Psi \right)^{-\frac{t}{2}}$, $t \in [0, 1]$.

4.2.

(a) Let $\Omega(\mathbb{R}^{2n})$ denote the space of linear symplectic forms on $\mathbb{R}^{2n}$. Show that

$$\Omega(\mathbb{R}^{2n}) \cong \mathrm{GL}(2n, \mathbb{R}) / \mathrm{Sp}(2n).$$

(b) Deduce that $\Omega(\mathbb{R}^{2n})$ is homotopy equivalent to $\mathrm{O}(2n)/\mathrm{U}(n)$.

Hint: Use a similar strategy as in Exercise 4.1 (c).

(c) Let $\mathcal{J}(\mathbb{R}^{2n})$ denote the space of linear complex structures on $\mathbb{R}^{2n}$. Show that

$$\mathcal{J}(\mathbb{R}^{2n}) \cong \mathrm{GL}(2n, \mathbb{R}) / \mathrm{GL}(n, \mathbb{C}).$$

(d) Show that $\mathrm{GL}(n, \mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{U}(n)$ and that $\mathcal{J}(\mathbb{R}^{2n})$ is homotopy equivalent to $\mathrm{O}(2n)/\mathrm{U}(n)$. In particular, $\Omega(\mathbb{R}^{2n})$ and $\mathcal{J}(\mathbb{R}^{2n})$ are homotopy equivalent.

***4.3.**

(a) Show that any co-oriented hypersurface $\Sigma \subset \mathbb{R}^3$ inherits an almost complex structure from the vector product as follows. Let $\nu\colon \Sigma \to S^2$ be the Gauss map. Then

$$J_x(u) := \nu(x) \times u$$

is an almost complex structure.

(b) Show that J is compatible with the symplectic form

$$\omega_x(v, w) = \nu(x) \cdot (v \times w)$$

(see Exercise 1.3).

(c) Show that every co-oriented hypersurface $M \subset \mathbb{R}^7$ also carries an almost complex structure.

(d) Give an example of an almost complex manifold that does not admit a symplectic structure.