The most important exercises are marked with an asterisk *.

5.1. Let $\omega \in \Omega^2(M)$ be a non-degenerate 2-form and $J \in \mathcal{J}_c(M,\omega)$. Let ∇ denote the Levi-Civita connection associated to the Riemannian metric $g_J(v,w) := \omega(v, Jw)$.

(a) Show that for any $X \in \Gamma(TM)$, we have

$$(\nabla_X J)J + J(\nabla_X J) = 0.$$

(b) Let $X, Y, Z \in \Gamma(TM)$ be three vector fields. Show that

$$g_J((\nabla_X J)Y, Z) + g_J(Y, (\nabla_X J)Z) = 0.$$

(c) Show that

$$\mathrm{d}\omega = g_J((\nabla_X J)Y, Z) + g_J((\nabla_Y J)Z, X) + g_J((\nabla_Z J)X, Y).$$

5.2. Let $\omega \in \Omega^2(M)$, $J \in \mathcal{J}_c(M,\omega)$, g_J and ∇ be as above. Show that the following are equivalent:

(i) $\nabla J = 0$

(ii) J is integrable and ω is closed.

***5.3.** Let $B(r) \subset \mathbb{R}^2$ denote the open disc of radius r. We use the coordinates x_1, y_1, x_2, y_2 and the symplectic form $\mathrm{d}y_1 \wedge \mathrm{d}x_1 + \mathrm{d}y_2 \wedge \mathrm{d}x_2$ on $\mathbb{R}^4$. Consider the product $B(r) \times B\left(\frac{1}{r}\right) \subset \mathbb{R}^4$.

(a) Show that there exists a volume preserving diffeomorphism

$$\psi \colon B(1) \times B(1) \to B(r) \times B\left(\frac{1}{r}\right)$$

for any $r > 0$.

(b) Let c be symplectic capacity in dimension 4. Show that

$$c\left(B(r) \times B\left(\frac{1}{r}\right), \omega_{\mathrm{std}}\right) \to 0$$

as $r \to 0$.

(c) Let $0 < r_1 \leq r_2$ and $0 < s_1 \leq s_2$. Show that there exists a symplectic diffeomorphism

$$\varphi\colon B(r_1) \times B(r_2) \to B(s_1) \times B(s_2)$$

if and only if $r_1 = s_1$ and $r_2 = s_2$.

Hint: You may use the fact that a symplectic capacity exists.

Remark: The generalization of (c) to the product of n open symplectic 2-balls in $\mathbb{R}^{2n}$ is true. The proof is more subtle and needs more machinery (e.g. symplectic homology).

***5.4.** Given a linear subspace $W \subset \mathbb{R}^{2n}$, its *symplectic complement* is defined by

$$W^{\perp} = \{v \in \mathbb{R}^{2n} \,|\, \omega_{\mathrm{std}}(v, w) = 0 \text{ for all } w \in W\}.$$

The subspace W is called *isotropic* if $W \subset W^{\perp}$.

(a) Show that $(W^{\perp})^{\perp} = W$ and $\dim W^{\perp} = \dim \mathbb{R}^{2n} - \dim W$.

(b) Show that if W is isotropic then $\dim W \leq n$.

(c) Let c be a symplectic capacity. Let $\Omega \subset \mathbb{R}^{2n}$ be an open bounded set containing 0 and $W \subset \mathbb{R}^{2n}$ a linear subspace of codimension 2. Show that

$$c(\Omega + W) = +\infty$$

if $W^{\perp}$ is isotropic. Here,

$$\Omega + W = \{x + w \in \mathbb{R}^{2n} \,|\, x \in \Omega, w \in W\}.$$

(d) Let $\Omega \subset \mathbb{R}^{2n}$ and $W \subset \mathbb{R}^{2n}$ be as above. Show that

$$0 < c(\Omega + W) < +\infty$$

if $W^{\perp}$ is not isotropic.