The most important exercises are marked with an asterisk *.

6.1.

(a) Let (V, ω) be a symplectic vector space and $g\colon V \times V \to \mathbb{R}$ be an inner product. Show that there exists a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ that is orthogonal with respect to g. Moreover, this basis can be chosen so that $g(e_j, e_j) = g(f_j, f_j)$.

Hint: Consider $\mathbb{R}^{2n}$ with the standard inner product and a linear symplectic form ω. Use an orthonormal basis $z_1, \ldots, z_n \in \mathbb{C}^n$ of eigenvectors of the skew-symmetric matrix A representing ω.

(b) Let g be an inner product on $\mathbb{R}^{2n}$ and consider the ellipsoid

$$E(g) = \left\{ w \in \mathbb{R}^{2n} \,|\, g(w, w) < 1 \right\}.$$

Show that there exists a symplectic linear matrix $A \in \mathrm{Sp}(2n)$ and an n-tuple $\boldsymbol{r} = (r_1, \ldots, r_n)$ with $0 < r_1 \leq \cdots \leq r_n$ and such that $AE = E(\boldsymbol{r})$, where

$$E(\boldsymbol{r}) = \left\{ (x, y) \in \mathbb{R}^{2n} \,\middle|\, \sum_{j=1}^{2n} \frac{x_j^2 + y_j^2}{r_j^2} < 1 \right\}.$$

(c) Show that the numbers $r_1, \ldots, r_n$ are uniquely determined by $E(g)$.

Hint: Suppose $E(\boldsymbol{r})$ and $E(\boldsymbol{s})$ are related by $A \in \mathrm{Sp}(2n)$. Show that $J_0 \operatorname{diag}(\frac{1}{r_1^2}, \ldots, \frac{1}{r_n^2})$ is similar to $J_0 \operatorname{diag}(\frac{1}{s_1^2}, \ldots, \frac{1}{s_n^2})$ and compare the eigenvalues.

(d) Interpret the result for $n = 1$.

6.2. Let $E \subset \mathbb{R}^{2n}$ be an ellipsoid centered at 0. Show that there exists $A \in \mathrm{GL}(2n, \mathbb{R})$ such that $A^*\omega_{\mathrm{std}} = -\omega_{\mathrm{std}}$ and $A(E) = E$.

***6.3.** Let $\psi_n\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a sequence of continuous maps converging to a homeomorphism $\psi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, uniformly on compact sets. Let $E \subset \mathbb{R}^{2n}$ be an ellipsoid centered at 0.

(a) Show that for any $\lambda < 1$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\psi_n(\lambda E) \subset \psi(E).$$

(b) Show that for any $\mu > 1$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\psi(E) \subset \psi_n(\mu E).$$

Hint: Consider maps $\phi_n \colon \mu\,\partial E \to S^{2n-1}$ obtained by normalizing $\psi^{-1} \circ \psi_n$ and study their degree.

(c) Deduce that if ψ_n preserve the capacity of all ellipsoids, then also ψ preserves the capacity of all ellipsoids.

***6.4.** Deduce from Exercise 6.3 that $\mathrm{Symp}(\mathbb{R}^{2n})$ is C^0-closed in $\mathrm{Diff}(\mathbb{R}^{2n})$.