The most important exercises are marked with an asterisk *.

7.1. Consider two functions $F, G \in C^\infty(\mathbb{R}^{2n})$. The Poisson bracket of F and G is defined by

$$\{F, G\} := \sum_{j=1}^{n} \left(\frac{\partial F}{\partial x_j} \frac{\partial G}{\partial y_j} - \frac{\partial F}{\partial y_j} \frac{\partial G}{\partial x_j} \right) \in C^\infty(\mathbb{R}^{2n}).$$

***(a)** Show that $\{F, G\} = \omega_{\mathrm{std}}(X^G, X^F)$.

***(b)** The function F is called an *integral* of the Hamiltonian differential equation

$$\dot{x}(t) = X^G(x(t))$$

if F is constant along its solutions $x(t)$. Show that F is an *integral* of the Hamiltonian differential equation associated to G if and only if $\{F, G\} = 0$.

***(c)** Show that a diffeomorphism $\psi \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism if and only if

$$\{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$$

for any functions $F, G \in C^\infty(M)$.

(d) Let X and Y be two symplectic vector fields. Show that $[X, Y]$ is a Hamiltonian vector field.

Hint: The bracket of two vector fields X and Y is defined by

$$[X, Y] = \nabla_X Y - \nabla_Y X.$$

Use the formula $\iota_{[X,Y]}\omega = \mathcal{L}_X(\iota_Y \omega) - \iota_Y(\mathcal{L}_X \omega)$ to show that $\iota_{[X,Y]}\omega_{\mathrm{std}}$ is exact.

(e) Show that

$$[X^F, X^G] = X^{\{G,F\}}.$$

***7.2.** Let c be a symplectic capacity. Define

$$\check{c}(M, \omega) := \sup\{c(U, \omega) \,|\, U \subset M \text{ open}, \overline{U} \subset M \backslash \partial M\}.$$

We always have $\check{c} \leq c$. The capacity c is called *inner regular* if $c = \check{c}$. Show:

(a) Corrected version!!! The measurement $\check{c}$ does satisfy the conformality and non-triviality axiom, but it is not necessarily a symplectic capacity.

Remark: Changing the definition of $\check{c}$ by taking supremum only over $c(U,\omega)$ where $\overline{U}$ is in addition compact, will result in an actual symplectic capacity.

(b) If d is any inner regular symplectic capacity with $d \leq c$, then $d \leq \check{c}$.

(c) The Gromov-width $D(M,\omega)$ is inner regular.

(d) The Hofer-Zehnder capacity c_0 is inner regular.

***7.3.** What is the biggest symplectic capacity?

7.4. Let $H \in C^\infty(\mathbb{R} \times \mathbb{R}^{2n})$ be a Hamiltonian function that is 1-periodic: $H_t = H_{t+1}$ for any t. On a loop $z \in C^\infty(S^1, \mathbb{R}^{2n})$, the action functional takes the value

$$\mathcal{A}_H(z) = \int_0^1 \frac{1}{2}\langle -J_0\dot{z}(t), z(t)\rangle \, \mathrm{d}t - \int_0^1 H_t(z(t)) \, \mathrm{d}t.$$

Show that this coincides with the physicist's action functional, namely for a loop $z(t) = (x(t), y(t))$ we have

$$\mathcal{A}_H(z) = \int_0^1 \langle y(t), \dot{x}(t)\rangle \, \mathrm{d}t - \int_0^1 H_t(z(t)) \, \mathrm{d}t.$$

In other words, $\mathcal{A}_H(z)$ is the integral of the *action 1-form*

$$\lambda_H := \sum_{j=1}^n y_j \mathrm{d}x_j - H\mathrm{d}t$$

along the loop z.

Hint: Integration by parts.