The most important exercises are marked with an asterisk *.

***8.1.** Let $N \in \mathbb{N}$, $\pi < a < 2\pi$ and consider the quadratic Hamiltonian $Q\colon \mathbb{R}^{2n} \to \mathbb{R}$ defined by

$$Q(x, y) = a(x_1^2 + y_1^2) + \frac{a}{N^2} \sum_{j=2}^{n} (x_j^2 + y_j^2).$$

Show that there are no 1-periodic solutions to $\dot{z} = X^Q(z)$ except the constant solution $z = 0$.

***8.2.** Let E be a Hilbert space and $f \in C^1(E, \mathbb{R})$. A subset $R \subset E$ is called a mountain range for f, if

- $E \backslash R$ is disconnected,
- $\alpha := \inf_R f > -\infty$,
- and on every component of $E \backslash R$, the function f attains a value strictly less than α.

Prove the Mountain Pass Lemma: Assume that f satisfies the Palais-Smale condition and assume that the gradient equation

$$\dot{x} = -\nabla f(x)$$

generates a global flow on E. Then for any mountain range $R \subset E$, the function f has a critical point $x \in E$ satisfying $f(x) \geq \alpha$.

8.3. Show that H^1 is a Hilbert space. You should use the fact that $L^2(S^1)$ is a Hilbert space and the definition of H^1 using Fourier series.

Hint: Given a Cauchy sequence $x^n \in H^1$, consider the sequences x^n and its weak derivative $y^n := \dot{(x^n)}$ as Cauchy sequences in $L^2(S^1)$.

8.4. If you don't know the Fourier series representation for elements in $L^2(S^1)$ and you are interested in it, read it up for example in sections 1.1 and 1.2 in `https://math.iisc.ac.in/~veluma/fourier.pdf`