The most important exercises are marked with an asterisk *.

***9.1.** Let $H\colon [0,1]\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ be a smooth compactly supported Hamiltonian function on $\mathbb{R}^{2n}$. Let $x(t)$ be a 1-periodic solution to $\dot{x}(t)=X^H(x(t))$. The goal of this exercise is to show that the action

$$\mathcal{A}_H(x)=\int_0^1 \frac{1}{2}\langle -J_0\dot{x}(t),x(t)\rangle-\int_0^1 H_t(x(t))\,\mathrm{d}t.$$

only depends on $x(0)$ and ψ_1^H and not on H.

(a) Let H and K be Hamiltonians as above and assume $\psi_1^H=\psi_1^K$. Consider the piecewise smooth path $t\mapsto\varphi_t$ defined by

$$\varphi_t=\begin{cases}\psi_t^H & t\in[0,1],\\ \psi_{2-t}^K & t\in[1,2].\end{cases}$$

Let $x_0\in\mathbb{R}^{2n}$ and define $\Delta(x_0)$ to be the action of the loop $x(t)=\varphi_t(x_0)$:

$$\Delta(x_0)=\int_0^2\frac{1}{2}\langle -J_0\dot{x}(t),x(t)\rangle\,\mathrm{d}t-\int_0^1 H_t(x(t))\,\mathrm{d}t+\int_1^2 K_{2-t}(x(t))\,\mathrm{d}t.$$

Show that

$$\Delta(x_0)=\mathcal{A}_H(\psi_t^H(x_0))-\mathcal{A}_K(\psi_t^K(x_0))$$

if $\psi_1^H(x_0)=x_0$.

(b) Show that Δ is differentiable and $\mathrm{d}\Delta_x=0$.

(c) Conclude that

$$\mathcal{A}_H(\psi_t^H(x_0))=\mathcal{A}_K(\psi_t^K(x_0))$$

if $\psi_1^H(x_0)=x_0$.

We can therefore define

$$\mathcal{A}(x,\varphi):=\mathcal{A}_H(\psi_t^H(x))$$

for $\varphi=\psi_1^H\in\mathrm{Ham}(\mathbb{R}^{2n})$ and any fixed point x of φ.

***9.2.** Let $\vartheta\in\mathrm{Symp}(\mathbb{R}^{2n})$, $\varphi\in\mathrm{Ham}_c(\mathbb{R}^{2n})$ a Hamiltonian diffeomorphism generated by a compactly supported Hamiltonian and $x\in\mathbb{R}^{2n}$ a fixed point of φ. Show that $\mathcal{A}(\vartheta(x),\vartheta\varphi\vartheta^{-1})=\mathcal{A}(x,\varphi)$.

Hint: The following exercises are useful: Exercises 2.2., 3.2(b) and 7.4.