The most important exercises are marked with an asterisk *.

1.1. Let $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth Hamiltonian function. Suppose H is compactly supported and let Φ_t be the associated Hamiltonian flow.

(a) Consider the Hamiltonian vector field

$$X(q,p) = \begin{pmatrix} \frac{\partial H}{\partial p}(q,p) \\ -\frac{\partial H}{\partial q}(q,p) \end{pmatrix}.$$

Show that $\operatorname{div} X = 0$.

Solution. By definition, the divergence of X is given by

$$\operatorname{div} X := \frac{\partial}{\partial q} X_q + \frac{\partial}{\partial p} X_p.$$

Substituting the definition of X, we get:

$$\operatorname{div} X = \frac{\partial^2 H}{\partial q \partial p} - \frac{\partial^2 H}{\partial p \partial q}.$$

Since H is smooth by assumption, it follows that

$$\frac{\partial^2 H}{\partial q \partial p} = \frac{\partial^2 H}{\partial p \partial q}$$

and thus $\operatorname{div} X = 0$.

(b) Let $Y\colon \mathbb{R}^m \to \mathbb{R}^m$ be any smooth vector field with $\operatorname{div} Y = 0$. Suppose $\Psi_t\colon \mathbb{R}^m \to \mathbb{R}^m$ is its flow, i.e.

$$\forall x \in \mathbb{R}^m\colon \qquad \frac{\mathrm{d}}{\mathrm{d}t}\Psi_t(x) = Y(\Psi_t(x)), \qquad \Psi_0 = \mathrm{id}\,. \tag{1}$$

Show that

$$\det\left(\mathrm{d}\Psi_t(x)\right) = 1 \qquad \text{for all } x \in \mathbb{R}^m \text{ and } t \in \mathbb{R}.$$

Hint: Jacobi's formula on how to express a derivative of a determinant might be helpful.

Solution. Note that, by definition of a flow, it holds: $\Psi_0 = \mathrm{id}$. Therefore, $\mathrm{d}\Psi_0(x) = \mathrm{id}$ and thus $\det(\mathrm{d}\Psi_0) = 1$. The idea now is to show that $\det(\mathrm{d}\Psi_t)$ does not change as we change t. We do this by showing that

$$\frac{\mathrm{d}}{\mathrm{d}t}\det(\mathrm{d}\Psi_t) = 0.$$

Indeed, by the Jacobi formula, we have:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\det(\mathrm{d}\Psi_t) &= \det(\mathrm{d}\Psi_t)\cdot\mathrm{tr}\left((\mathrm{d}\Psi_t)^{-1}\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\Psi_t)\right) \\
&= \det(\mathrm{d}\Psi_t)\cdot\mathrm{tr}\left((\mathrm{d}\Psi_t)^{-1}\cdot\mathrm{d}\left(\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t\right)\right) \\
&= \det(\mathrm{d}\Psi_t)\cdot\mathrm{tr}\big((\mathrm{d}\Psi_t)^{-1}\cdot\mathrm{d}\left(Y(\Psi_t)\right)\big) && \text{(by equation (1))} \\
&= \det(\mathrm{d}\Psi_t)\cdot\mathrm{tr}\big((\mathrm{d}\Psi_t)^{-1}\cdot\mathrm{d}Y(\Psi_t)\cdot\mathrm{d}\Psi_t\big) && \text{(by the chain rule)} \\
&= \det(\mathrm{d}\Psi_t)\cdot\mathrm{tr}\big((\mathrm{d}\Psi_t)^{-1}\cdot\mathrm{d}\Psi_t\cdot\mathrm{d}Y(\Psi_t)\big) && (\mathrm{tr}(AB)=\mathrm{tr}(BA)) \\
&= \det(\mathrm{d}\Psi_t)\cdot\mathrm{tr}(\mathrm{d}Y(\Psi_t)) \\
&= \det(\mathrm{d}\Psi_t)\cdot\mathrm{div}\,Y(\Psi_t) \\
&= 0 && (\text{since } \mathrm{div}\,Y=0)
\end{aligned}$$

(c) Deduce from parts (a) and (b) that Φ_t is volume-preserving, i.e. for each bounded open domain $U\subseteq\mathbb{R}^{2n}$

$$\mathrm{vol}(U)=\mathrm{vol}(\Phi_t(U)).$$

This is the Liouville Theorem.

Solution. We will also denote by vol the volume form. We have:

$$\begin{aligned}
\mathrm{vol}(\Phi_t(U)) &= \int_{\Phi_t(U)}\mathrm{vol} && \text{(by definition)} \\
&= \int_U|\det(\mathrm{d}\Phi_t)|\,\mathrm{vol} && \text{(by change of variables)} \\
&= \int_U\mathrm{vol} && \text{(by part (b))} \\
&= \mathrm{vol}(U).
\end{aligned}$$

This completes the proof.

***1.2.** Consider the standard volume form on $\mathbb{R}^{2n}$:

$$\mathrm{vol}=\mathrm{d}p_1\wedge\mathrm{d}q_1\wedge\cdots\wedge\mathrm{d}p_n\wedge\mathrm{d}q_n.$$

Show that

$$\mathrm{vol}=\frac{\omega_{\mathrm{std}}^{\wedge n}}{n!},$$

where

$$\omega_{\text{std}} = \sum_{i=1}^{n} \mathrm{d}p_i \wedge \mathrm{d}q_i.$$

Deduce that any symplectomorphism $\Phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is volume-preserving.

Solution. Wedging ω_{std} with itself n times, we get:

$$\begin{aligned}
\omega_{\text{std}}^{\wedge n} &= \left(\sum_{k_1=1}^{n} \mathrm{d}p_{k_1} \wedge \mathrm{d}q_{k_1}\right) \wedge \cdots \wedge \left(\sum_{k_n=1}^{n} \mathrm{d}p_{k_n} \wedge \mathrm{d}q_{k_n}\right) \\
&= \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} \mathrm{d}p_{k_1} \wedge \mathrm{d}q_{k_1} \wedge \cdots \wedge \mathrm{d}p_{k_n} \wedge \mathrm{d}q_{k_n} \\
&\overset{(\star)}{=} \sum_{\sigma \in S_n} \mathrm{d}p_{\sigma(1)} \wedge \mathrm{d}q_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}p_{\sigma(n)} \wedge \mathrm{d}q_{\sigma(n)} \\
&\overset{(\star\star)}{=} \sum_{\sigma \in S_n} (-1)^{2\cdot\operatorname{sgn}(\sigma)} \cdot \mathrm{d}p_1 \wedge \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}p_n \wedge \mathrm{d}q_n \\
&\overset{(\star\star\star)}{=} n! \cdot \mathrm{d}p_1 \wedge \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}p_n \wedge \mathrm{d}q_n \\
&= n! \operatorname{vol}.
\end{aligned}$$

In equality $(\star)$ we use the fact that $\mathrm{d}p_{k_1} \wedge \mathrm{d}q_{k_1} \wedge \cdots \wedge \mathrm{d}p_{k_n} \wedge \mathrm{d}q_{k_n}$ vanishes if $k_i = k_j$ for some $i \neq j$ and we write the remaining terms in terms of permutations. Here, S_n denotes the set of all permutations of n elements.

To get $(\star\star)$, we use the fact that to go from $\mathrm{d}p_{\sigma(1)} \wedge \mathrm{d}q_{\sigma(1)} \wedge \cdots \wedge \mathrm{d}p_{\sigma(n)} \wedge q_{\sigma(n)}$ to $\mathrm{d}p_1 \wedge \mathrm{d}q_1 \wedge \cdots \wedge \mathrm{d}p_n \wedge \mathrm{d}q_n$ for some permutation $\sigma \in S_n$, we need to make an even number of swaps of adjacent terms. This gives us the factor $(-1)^{2\cdot\operatorname{sgn}(\sigma)}$.

Lastly, we get the factor of $n!$ in $(\star\star\star)$ because there are $n!$ permutations of n elements. This shows $\operatorname{vol} = \frac{\omega_{\text{std}}^{\wedge n}}{n!}$.

By definition, if $\Phi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism, then $\Phi^*\omega_{\text{std}} = \omega_{\text{std}}$. By the previous part it thus follows:

$$\Phi^* \operatorname{vol} = \Phi^*\left(\frac{\omega_{\text{std}}^{\wedge n}}{n!}\right) = \frac{1}{n!}\Phi^*\left(\omega_{\text{std}}^{\wedge n}\right) = \frac{1}{n!}\Phi^*\omega_{\text{std}} \wedge \cdots \wedge \Phi^*\omega_{\text{std}} = \frac{1}{n!}\omega_{\text{std}}^{\wedge n} = \operatorname{vol}.$$

***1.3.** Let $\Sigma \subseteq \mathbb{R}^3$ be a surface, i.e. a smooth 2-dimensional submanifold, and let $\nu\colon \Sigma \to \mathbb{R}^3$ be a co-orientation of Σ, i.e. a smooth unit normal vector field. We define the 2-form ω on Σ by

$$\omega_x(v, w) \coloneqq \nu(x) \cdot (v \times w) \qquad \forall x \in \Sigma,\ v, w \in T_x\Sigma,$$

where $\cdot$ denotes the Euclidean inner product and $\times$ denotes the cross product.

(a) Show that ω is a symplectic form.

Solution. To show that ω is a symplectic form, we need to show it is a closed non-degenerate 2-form.

To show ω is a differential form, we need to show that ω depends smoothly on x and that ω_x it is bilinear and antisymmetric: ν is a smooth function and therefore ω depends smoothly on x; the Euclidean inner product and the cross product are bilinear and thus so is ω_x; antisymmetry follows from the antisymmetry of the cross product.

We showed that ω is a 2-form. Its exterior derivative is thus of degree 3. Since Σ is 2-dimensional, $\mathrm{d}\omega$ vanishes. This shows ω is closed.

To show ω is non-degenerate, we need to show that for any non-zero vector $v \in T_x\Sigma$, there exists another vector $w \in T_x\Sigma$ such that $\omega_x(v, w) \neq 0$. Let $v \in T_x\Sigma$ be a non-zero vector. We claim that $w = \nu(x) \times v$ has the required property. Using the Grassmann identity:

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c,$$

we have:

$$\begin{aligned}\omega_x(v, \nu(x) \times v) &= \nu(x) \cdot (v \cdot (\nu(x) \times v)) \\ &= \nu(x) \cdot ((v \times v)\nu(x) - 0) \\ &= \|v\|^2 \neq 0.\end{aligned}$$

This shows that ω is non-degenerate and completes the proof that ω is a symplectic form.

(b) Give a formula for ω in the case of the 2-sphere:

$$S^2 := \{x \in \mathbb{R}^3 \mid |x| = 1\}.$$

Solution. For the 2-sphere S^2, a unit normal vector field is given by:

$$\nu\colon S^2 \to \mathbb{R}^3, \qquad \nu(x) := x.$$

The form ω is therefore given by:

$$\omega_x(v, w) = x \cdot (v \times w) = \det \begin{pmatrix} | & | & | \\ x & v & w \\ | & | & | \end{pmatrix},$$

for all $x \in S^2$ and $v, w \in T_xS^2 = \{v \in \mathbb{R}^3 \mid x \cdot v = 0\}$.

***1.4.** Let (M, ω) be a $2n$-dimensional symplectic manifold. Show that $\omega^{\wedge n}$ is a volume form, i.e. that $\omega^{\wedge n}$ is a nowhere vanishing form of top degree.

Note that would imply that M is canonically oriented. The form $\frac{\omega^{\wedge n}}{n!}$ is called the *symplectic volume*.

Solution. Recall that, by definition, for every point $x \in M$ there exists a neighbourhood $U \subset M$ of x and a diffeomorphism $\varphi\colon U \to \varphi(U) \subset \mathbb{R}^{2n}$ such that $\varphi^*\omega_{\text{std}} = \omega$. In Exercise 1.2, we saw that $\omega_{\text{std}}^{\wedge n}$ is a volume form on $\mathbb{R}^{2n}$. Since φ is a diffeomorphism, it follows that the form

$$\omega^{\wedge n} = (\varphi^*\omega_{\text{std}}) \wedge \cdots \wedge (\varphi^*\omega_{\text{std}}) = \varphi^*\left(\omega_{\text{std}}^{\wedge n}\right)$$

is also a volume form on M.