The most important exercises are marked with an asterisk *.

***2.1.** Let (M, ω) be a closed symplectic manifold of positive dimension. Show that ω is not exact.

Solution. Assume M is of dimension $2n$. We saw that $\Omega \coloneqq \omega^{\wedge n}$ is a volume form on M and hence does not vanish anywhere. We equip M with the orientation induced by Ω. It follows that the integral $\int_M \omega^{\wedge n}$ is positive.

Assume for contradiction that ω is exact and let α be a 1-form such that $\omega = \mathrm{d}\alpha$. It follows from Leibniz rule that

$$\mathrm{d}(\alpha \wedge \omega^{\wedge(n-1)}) = \omega^{\wedge n}.$$

Stokes theorem now implies that

$$\int_M \omega^{\wedge n} = \int_{\partial M} \alpha \wedge \omega^{\wedge(n-1)},$$

where ∂M denotes the boundary of M. Since the left-hand side is positive, it follows that the boundary ∂M is non-empty. This contradicts the assumption that M is a closed (i.e. compact without boundary) manifold.

***2.2.** Let (M, ω) be a symplectic manifold, $H, K \colon [0,1] \times M \to \mathbb{R}$ be two smooth Hamiltonian functions and $\chi \in \mathrm{Symp}(M, \omega)$.

(a) Show that $\psi_t^H \circ \psi_t^K$ is generated by

$$(H\#K)_t = H_t + K_t \circ (\psi_t^H)^{-1}.$$

Solution. We compute

$$\begin{aligned}
\mathrm{d}\,(H\#K)_t &= \mathrm{d}H_t + \mathrm{d}K_t \circ \mathrm{d}(\psi_t^H)^{-1} \\
&= -\omega(X_t^H, -) - \omega\left(X_t^K \circ \left(\psi_t^H\right)^{-1}, \mathrm{d}(\psi_t^H)^{-1}(-)\right) \\
&= -\omega(X_t^H, -) - \omega\left(\mathrm{d}\psi_t^H\left(X_t^K \circ \left(\psi_t^H\right)^{-1}\right), -\right) \\
&= -\omega\left(X_t^H + \left(\psi_t^H\right)_* (X_t^K), -\right),
\end{aligned}$$

hence $X_t^{H\#K} = X_t^H + \left(\psi_t^H\right)_* (X_t^K)$. In the third inequality we used that ψ_t^H is symplectic. Moreover,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^H \circ \psi_t^K\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^H\right) \circ \psi_t^K + \mathrm{d}\psi_t^H \left(\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^K\right) \\ &= X_t^H \circ \psi_t^H \circ \psi_t^K + \mathrm{d}\psi_t^H \left(X_t^K \circ \psi_t^K\right) \\ &= X_t^{H\#K} \circ (\psi_t^H \circ \psi_t^K)\end{aligned}$$

which proves the claim.

(b) Show that $\left(\psi_t^H\right)^{-1}$ is generated by

$$\overline{H}_t = -H_t \circ \psi_t^H.$$

Solution. We proceed as before:

$$\begin{aligned}\mathrm{d}\overline{H}_t &= -\mathrm{d}H_t \circ \mathrm{d}\psi_t^H \\ &= \omega\left(X_t^H \circ \psi_t^H, \mathrm{d}\psi_t^H(-)\right) \\ &= \omega\left(\mathrm{d}\left(\psi_t^H\right)^{-1}\left(X_t^H \circ \psi_t^H\right), -\right) \\ &= \omega\left(\left(\psi_t^H\right)^* (X_t^H), -\right),\end{aligned}$$

hence $X_t^{\overline{H}} = -\left(\psi_t\right)^* (X_t^H)$. On the other hand

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^H \circ \left(\psi_t^H\right)^{-1}\right) \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^H\right) \circ \left(\psi_t^H\right)^{-1} + \mathrm{d}\psi_t^H \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^H\right)^{-1}\right) \\ &= X_t^H + \mathrm{d}\psi_t^H \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^H\right)^{-1}\right),\end{aligned}$$

hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^H\right)^{-1} = -\mathrm{d}\left(\psi_t^H\right)^{-1}\left(X_t^H\right) = X_t^{\overline{H}} \circ \left(\psi_t^H\right)^{-1}.$$

(c) Show that $\chi^{-1}\psi_t^H \chi$ is generated by $H_t \circ \chi$.

Solution. We compute

$$\begin{aligned}\mathrm{d}(H_t \circ \chi) &= \mathrm{d}H_t \circ \mathrm{d}\chi \\ &= -\omega\left(X_t^H \circ \chi, \mathrm{d}\chi(-)\right) \\ &= -\omega\left(\mathrm{d}\chi^{-1}\left(X_t^H \circ \chi\right), -\right),\end{aligned}$$

hence $X_t^{H\circ\chi} = \chi^*(X_t^H)$. Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\chi^{-1}\psi_t^H\chi\right) &= \mathrm{d}\chi^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^H \circ \chi\right) \\ &= \chi^*\left(\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^H\right) = \chi^*(X_t^H \circ \psi_t^H) = X_t^{H\circ\chi} \circ \left(\chi^{-1}\psi_t^H\chi\right).\end{aligned}$$

(d) Deduce from parts (a), (b) and (c) that $\mathrm{Ham}(M,\omega)$ is a normal subgroup of $\mathrm{Symp}(M,\omega)$.

Solution. (a) and (b) show that $\mathrm{Ham}(M,\omega) \subset \mathrm{Symp}(M)$ is closed under composition and inverse. It is therefore a subgroup. (c) shows that $\mathrm{Ham}(M,\omega)$ is closed under conjugation by an element in $\mathrm{Symp}(M,\omega)$. $\mathrm{Ham}(M,\omega)$ is therefore normal in $\mathrm{Symp}(M,\omega)$.

2.3. Find an example of a symplectic manifold (M,ω) (without boundary) and a smooth function $H\colon [0,+\infty)\times M \to \mathbb{R}$ such that the domain $\mathcal{D}_H$ of the Hamiltonian flow ψ^H is not equal to $[0,+\infty)\times M$. This flow is by definition the flow of the time-dependent Hamiltonian vector field X^{H_t}, which is defined by $-\mathrm{d}H^t = \omega(X^{H_t},\cdot)$.

Find an example in which ψ_t^H is not surjective for some t.

Solution. Consider

$$M \coloneqq (0,+\infty)\times\mathbb{R}, \qquad \omega \coloneqq \omega_0, \qquad H\colon [0,+\infty)\times M \to \mathbb{R}, \qquad H(t,q,p) = -p.$$

We show that the domain of the Hamiltonian flow φ_H is

$$\mathcal{D}_H = \{(t,q,p) \mid t < q\}.$$

Indeed, Hamilton's equations for H are

$$\dot{q} = \frac{\partial H}{\partial p} = -1, \qquad \dot{p} = -\frac{\partial H}{\partial q} = 0.$$

The unique maximal solution of these equations starting at $(q,p)(0) = (q_0,p_0)$ is

$$(q,p)\colon [0,q_0) \to M, \qquad (q,p)(t) = (q_0 - t, p_0).$$

Since M is given by $M = (0,\infty)\times\mathbb{R}$ we cannot extend this solution beyond $t = q_0$.

Consider now

$$M \coloneqq (0,+\infty)\times\mathbb{R}, \qquad \omega \coloneqq \omega_0, \qquad H\colon [0,+\infty)\times M \to \mathbb{R}, \qquad H(t,q,p) = p.$$

Then we have

$$\psi_t^H\colon \mathcal{M} \to M, \qquad \psi_t^H(q_0,p_0) = (q_0 + t, p_0).$$

The map φ_t^H is not surjective for $t > 0$.

***2.4.** This problem is intended for students who are not yet familiar with the Lie derivative and Cartan's formula. Let $I \subset \mathbb{R}$ be an open interval, M a closed smooth manifold and $f\colon I \times M \to M$ a smooth function such that $f_s \in \mathrm{Diff}(M)$ for all $s \in I$. Let $k \geq 1$ and $\omega \in \Omega^k(M)$ a differential k-form on M.

The goal of this exercise is to prove Cartan's formula:

$$\frac{\mathrm{d}}{\mathrm{d}s} f_s^*\omega = f_s^*(\iota_{X_s}\mathrm{d}\omega + d\iota_{X_s}\omega),$$

where X_s is the time-dependent vector field on M defined by

$$\frac{\mathrm{d}}{\mathrm{d}s} f_s = X_s \circ f_s.$$

For $s \in I$ consider the linear map

$$T_s\colon \Omega^k(I \times M) \to \Omega^{k-1}(M),$$

defined by

$$(T_s\sigma)_x(w_1, \dots w_{k-1}) = \sigma_{(s,x)}((1,0), (0, w_1), \dots, (0, w_{k-1}))$$

for $x \in M$ and $w_1, \dots, w_{k-1} \in T_xM$. Here we use the decomposition

$$T_{(s,x)}(I \times M) \cong T_s\mathbb{R} \oplus T_xM \cong \mathbb{R} \oplus T_xM.$$

(a) Let $f \in C^\infty(I \times M)$ and $\alpha \in \Omega^{k-1}(M)$. Prove that for $\sigma = f\,\mathrm{d}s \wedge \pi^*\alpha$ we have

$$\frac{\mathrm{d}}{\mathrm{d}s} i_s^*\sigma = (T_s\mathrm{d} + \mathrm{d}T_s)\sigma,$$

where $i_s\colon M \to I \times M$, $i_s(x) = (s,x)$ and $\pi\colon I \times M \to M$, $\pi(s,x) = x$.

Solution. We show that both sides of the equality vanish.

Note that for $s_0 \in I$, we have $i_{s_0}^*(\mathrm{d}s) = \mathrm{d}(s \circ i_{s_0}) = \mathrm{d}s_0 = 0$. Therefore

$$i_{s_0}^*\sigma = (f \circ i_{s_0})\ \mathrm{d}(s \circ i_{s_0}) \wedge i_{s_0}^*\pi^*\alpha = 0$$

and thus $i_s^*\sigma = 0$. It follows that

$$\frac{\mathrm{d}}{\mathrm{d}s} i_s^*\sigma = 0.$$

To show the right-hand side vanishes, we note that the form $\mathrm{d}s$ doesn't vanish only if we feed it vectors spanned by $(1,0)$, whereas $\pi^*\alpha$ doesn't vanish at a point $(s,x) \in I \times M$ only on vectors of the form (v,w) for $w \in T_xM$, $w \neq 0$ (here, v can be any vector in T_sI). Therefore:

$$\begin{aligned}
&(T_s\mathrm{d}\sigma)_x(w_1,\dots,w_k)\\
&\quad= \mathrm{d}\sigma_{(s,x)}\Big((1,0),(0,w_1),\dots,(0,w_k)\Big)\\
&\quad= (\mathrm{d}f \wedge \mathrm{d}s \wedge \pi^*\alpha - f\mathrm{d}s \wedge \pi^*\mathrm{d}\alpha)_{(s,x)}\Big((1,0),(0,w_1),\dots,(0,w_k)\Big)\\
&\quad\overset{(\diamondsuit)}{=} -(\mathrm{d}f \wedge \pi^*\alpha)_{(s,x)}\Big((0,w_1),\dots,(0,w_k)\Big) - (f\ \pi^*\mathrm{d}\alpha)_{(s,x)}\Big((0,w_1),\dots,(0,w_k)\Big)\\
&\quad= -(\mathrm{d}f \wedge \pi^*\alpha)_{(s,x)}(\mathrm{d}i_s)_x(w_1,\dots,w_k) - (f\ \pi^*\mathrm{d}\alpha)_{(s,x)}(\mathrm{d}i_s)_x(w_1,\dots,w_k)\\
&\quad= -\Big(\mathrm{d}(f\circ i_s)\wedge\alpha + (f\circ i_s)\mathrm{d}\alpha\Big)_x(w_1,\dots,w_k)
\end{aligned}$$

We obtained equality $(\diamondsuit)$ by feeding $(1,0)$ to $\mathrm{d}s$ (this also introduced a minus sign in the first term, because of the 1-form $\mathrm{d}f$ in front of $\mathrm{d}s$). In the last equality, we used $\pi \circ i_s = \mathrm{id}$ to get rid of π^*.

Before calculating $\mathrm{d}T_s\sigma$, note that:

$$\begin{aligned}
&(T_s\sigma)_x(w_1,\dots,w_{k-1})\\
&\quad= \sigma_{s,x}\Big((w_1,\dots,w_{k-1})\Big)\\
&\quad= (f\ \mathrm{d}s \wedge \pi^*\alpha)_{(s,x)}\Big((1,0),(0,w_1),\dots,(0,w_{k-1})\Big)\\
&\quad\overset{(\spadesuit)}{=} (f\ \pi^*\alpha)_{(s,x)}\Big((0,w_1),\dots,(0,w_{k-1})\Big)\\
&\quad= (f\ \pi^*\alpha)_{(s,x)}(\mathrm{d}i_s)_x(w_1,\dots,w_{k-1})\\
&\quad\overset{(\clubsuit)}{=} ((f\circ i_s)\ \alpha)_x(w_1,\dots,w_{k-1}),
\end{aligned}$$

where in $(\spadesuit)$ we fed $(1,0)$ to $\mathrm{d}s$ as above, and in $(\clubsuit)$ we used $\pi \circ i_s = \mathrm{id}$. In the last equality, f became $f \circ i_s$ because in the end we switched from calculating the form at the point $(s,x) \in I \times M$ to the point $x \in M$ (and recall that f is a function on $I \times M$). Taking the exterior derivative of $T_s\sigma$ and using the above calculation, we get:

$$\mathrm{d}T_s\sigma = \mathrm{d}\Big((f\circ i_s)\ \alpha\Big) = \mathrm{d}(f\circ i_s)\wedge\alpha + (f\circ i_s)\mathrm{d}\alpha = -T_s\mathrm{d}\sigma.$$

In particular, both sides of the claimed equality are 0.

(b) Let $f \in C^\infty(I \times M)$ and $\beta \in \Omega^k(M)$. Prove that for $\sigma = f\,\pi^*\beta$, we have

$$\frac{\mathrm{d}}{\mathrm{d}s} i_s^*\sigma = (T_s\mathrm{d} + \mathrm{d}T_s)\sigma.$$

Solution. Firstly, note that since $\pi \circ i_s = \mathrm{id}$, we have $i_s^*\sigma = (i_s^* f)(\pi \circ i_s)^*\beta = (f \circ i_s)\beta$. Therefore, if we denote $f_s \coloneqq f \circ i_s$, we get

$$\frac{\mathrm{d}}{\mathrm{d}s} i_s^*\sigma = \left(\frac{\mathrm{d}}{\mathrm{d}s} f_s\right)\beta.$$

On the one hand, β is a form on M and thus $T_s\sigma = T_s(f\ \pi^*\beta)$ vanishes if we feed it the vector $(1,0)$. Thus we have $T_s\sigma = 0$, and consequently $\mathrm{d}T_s\sigma = 0$.

On the other hand, we have

$$\mathrm{d}\sigma = \mathrm{d}f \wedge \pi^*\beta + f\pi^*\mathrm{d}\beta$$

and applying T_s to $\mathrm{d}\sigma$, we get:

$$\begin{aligned}
&(T_s\mathrm{d}\sigma)_x(w_1,\dots,w_k)\\
&\quad = \Big(\mathrm{d}f \wedge \pi^*\beta + f\pi^*\mathrm{d}\beta\Big)_{(s,x)}\Big((1,0),(0,w_1),\dots,(0,w_k)\Big)\\
&\quad \overset{(\heartsuit)}{=} \mathrm{d}f_{(s,x)}\Big((1,0)\Big)\ \Big(\pi^*\beta\Big)_{(s,x)}\Big((0,w_1),\dots,(0,w_k)\Big)\\
&\qquad\quad + (f\pi^*\mathrm{d}\beta)_{(s,x)}\Big((1,0),(0,w_1),\dots,(0,w_k)\Big)\\
&\quad = \mathrm{d}f_{(s,x)}\Big((1,0)\Big)\ \Big(\pi^*\beta\Big)_{(s,x)}\Big((0,w_1),\dots,(0,w_k)\Big)\\
&\quad = \mathrm{d}f_{(s,x)}\Big((1,0)\Big)\ \beta_x(w_1,\dots,w_k)\\
&\quad \overset{(\blacktriangle)}{=} \mathrm{d}f_{(s,x)}\left(\frac{\partial}{\partial s}\right)\ \beta_x(w_1,\dots,w_k)\\
&\quad = \frac{\mathrm{d}}{\mathrm{d}s} f(s,x)\ \beta_x(w_1,\dots,w_k)\\
&\quad = \frac{\mathrm{d}}{\mathrm{d}s} (f\circ i_s)(x)\ \beta_x(w_1,\dots,w_k)\\
&\quad = \frac{\mathrm{d}}{\mathrm{d}s} f_s(x)\ \beta_x(w_1,\dots,w_k)\\
&\quad = \left(\left(\frac{\mathrm{d}}{\mathrm{d}s} f_s\right)\ \beta\right)_x (w_1,\dots,w_k)
\end{aligned}$$

where, in $(\heartsuit)$ we fed the vector $(1,0)$ to $\mathrm{d}f$ because, after writing out the definition of the wedge product and applying it to the given vectors, in all the other terms we would be feeding $(1,0)$ to $\pi^*\beta$ and in that case:

$$\pi^*\beta\Big((1,0),\dots\Big) = \beta\Big(\mathrm{d}\pi(1,0),\dots\Big) = \beta(0,\dots) = 0.$$

The second term in equality (♡) vanishes for the same reason, i.e. since $d\pi((1,0)) = 0$. In equality (▲), we rewrote the vector $(1,0)$ as $\frac{\partial}{\partial s}$ and then used the definition of the differential:

$$\mathrm{d}f_{(s,x)}\left(\frac{\partial}{\partial s}\right) = \frac{\mathrm{d}}{\mathrm{d}s} f(s,x).$$

The result follows.

(c) Deduce from parts (a) and (b) that

$$\frac{\mathrm{d}}{\mathrm{d}s} i_s^* \sigma = (T_s \mathrm{d} + \mathrm{d} T_s)\sigma$$

for every $\sigma \in \Omega^k(I \times M)$.

Solution. Every k-form σ on $I \times M$ is a linear combination of forms of the type $f\,\mathrm{d}s \wedge \pi^*\alpha$ as in (a) and forms of the type $f\,\pi^*\beta$ as in (b). The formula follows from the previous steps and linearity.

(d) Prove Cartan's formula.

Solution. Note that now $f\colon I \times M \to M$ denotes a smooth function such that $f_s = f \circ i_s \in \mathrm{Diff}(M)$ for all $s \in I$. We still denote by $i_s\colon M \to I \times M$ the map given by $i_s(x) = (s,x)$.

Using the previous part of the exercise, we get:

$$\begin{aligned}
&\left(\frac{\mathrm{d}}{\mathrm{d}s} f_s^* \omega\right)_x (w_1, \dots, w_k) \\
&\qquad = \left(\frac{\mathrm{d}}{\mathrm{d}s} (f \circ i_s)^* \omega\right)_x (w_1, \dots, w_k) \\
&\qquad = \left(\frac{\mathrm{d}}{\mathrm{d}s} i_s^* f^* \omega\right)_x (w_1, \dots, w_k) \\
&\qquad = \Big((T_s \mathrm{d} + \mathrm{d} T_s) f^* \omega\Big)_x (w_1, \dots, w_k) \qquad (1)
\end{aligned}$$

For the first term in line (1), using the definition of T_s, rewriting $(1,0)$ as $\frac{\partial}{\partial s}$

and using the definition of a differential as in the previous part, we get:

$$\begin{aligned}
&(T_s \mathrm{d} f^* \omega)_x(w_1, \dots, w_k) \\
&\quad = (f^* \mathrm{d}\omega)_{(s,x)}\Big((1,0), (0, w_1), \dots (0, w_k)\Big) \\
&\quad = (\mathrm{d}\omega)_{f(s,x)}\Big(\mathrm{d}f((1,0)), \mathrm{d}f((0,w_1)), \dots, \mathrm{d}f((0,w_k))\Big) \\
&\quad = (\mathrm{d}\omega)_{f(s,x)}\left(\frac{\mathrm{d}}{\mathrm{d}s} f,\ \mathrm{d}f((0,w_1)), \dots, \mathrm{d}f((0,w_k))\right) \\
&\quad = (\mathrm{d}\omega)_{f_s(x)}\left(\frac{\mathrm{d}}{\mathrm{d}s} f_s,\ \mathrm{d}f_s(w_1), \dots, \mathrm{d}f_s(w_k)\right) \\
&\quad = (\mathrm{d}\omega)_{f_s(x)}\Big(X_s \circ f_s,\ \mathrm{d}f(w_1), \dots, \mathrm{d}f(w_k)\Big) \\
&\quad = \Big(\iota_{X_s} \mathrm{d}\omega\Big)_{f_s(x)}\Big(\mathrm{d}f(w_1), \dots, \mathrm{d}f(w_k)\Big) \\
&\quad = \Big(f_s^*(\iota_{X_s} \mathrm{d}\omega)\Big)_x (w_1, \dots, w_k).
\end{aligned}$$

In the following calculation of second term in line (1), by the notation $\hat{w}_i$, we mean that the vector field w_i is omitted. Using the definition of the exterior derivative and same steps in the above calculations, we obtain:

$$
\begin{aligned}
&\Big(\mathrm{d}T_s f^*\omega\Big)_x(w_1,\dots,w_k)\\
&\quad= \sum_{i=1}^{k}(-1)^i w_i\Big((T_s f^*\omega)_x(w_1,\dots,\hat{w}_i,\dots,w_k)\Big)\\
&\qquad + \sum_{1\le i<j\le k}(-1)^{i+j}(T_s f^*\omega)_x\Big([w_i,w_j],w_1,\dots,\hat{w}_i,\dots,w_k\Big)\\
&\quad= \sum_{i=1}^{k}(-1)^i w_i\Big((f^*\omega)_{(s,x)}\Big((1,0),(0,w_1),\dots,(0,\hat{w}_i),\dots,(0,w_k)\Big)\Big)\\
&\qquad + \sum_{1\le i<j\le k}(-1)^{i+j}(f^*\omega)_{(s,x)}\Big((1,0),(0,[w_i,w_j]),(0,w_1),\dots,(0,\hat{w}_i),\dots,(0,w_k)\Big)\\
&\quad= \sum_{i=1}^{k}(-1)^i w_i\Big(\omega_{f(s,x)}\Big(\mathrm{d}f(1,0),\mathrm{d}f(0,w_1),\dots,\mathrm{d}f(\hat{0,w_i}),\dots,\mathrm{d}f(0,w_k)\Big)\Big)\\
&\qquad + \sum_{1\le i<j\le k}(-1)^{i+j}\omega_{f(s,x)}\Big(\mathrm{d}f(1,0),\mathrm{d}f(0,[w_i,w_j]),\mathrm{d}f(0,w_1),\dots,\mathrm{d}f(\hat{0,w_i}),\dots,\mathrm{d}f(0,w_k)\Big)\\
&\quad= \sum_{i=1}^{k}(-1)^i w_i\left(\omega_{f_s(x)}\left(\frac{\mathrm{d}}{\mathrm{d}s}f_s,\mathrm{d}f_s(w_1),\dots,\mathrm{d}\hat{f_s(w_i)},\dots,\mathrm{d}f_s(w_k)\right)\right)\\
&\qquad + \sum_{1\le i<j\le k}(-1)^{i+j}\omega_{f_s(x)}\left(\frac{\mathrm{d}}{\mathrm{d}s}f_s,\mathrm{d}f_s([w_i,w_j]),\mathrm{d}f_s(w_1),\dots,\mathrm{d}\hat{f_s(w_i)},\dots,\mathrm{d}f_s(w_k)\right)\\
&\quad= \sum_{i=1}^{k}(-1)^i w_i\left(\omega_{f_s(x)}\Big(X_s\circ f_s,\ \mathrm{d}f_s(w_1),\dots,\mathrm{d}\hat{f_s(w_i)},\dots,\mathrm{d}f_s(w_k)\Big)\right)\\
&\qquad + \sum_{1\le i<j\le k}(-1)^{i+j}\omega_{f_s(x)}\Big(X_s\circ f_s,\ \mathrm{d}f_s([w_i,w_j]),\mathrm{d}f_s(w_1),\dots,\mathrm{d}\hat{f_s(w_i)},\dots,\mathrm{d}f_s(w_k)\Big)\\
&\quad= \sum_{i=1}^{k}(-1)^i w_i\left(\iota_{X_s}\omega_{f_s(x)}\Big(\mathrm{d}f_s(w_1),\dots,\mathrm{d}\hat{f_s(w_i)},\dots,\mathrm{d}f_s(w_k)\Big)\right)\\
&\qquad + \sum_{1\le i<j\le k}(-1)^{i+j}\iota_{X_s}\omega_{f_s(x)}\Big(\mathrm{d}f_s([w_i,w_j]),\mathrm{d}f_s(w_1),\dots,\mathrm{d}\hat{f_s(w_i)},\dots,\mathrm{d}f_s(w_k)\Big)\\
&\quad= \sum_{i=1}^{k}(-1)^i w_i\left((f_s^*\iota_{X_s}\omega)_x\,(w_1,\dots,\hat{w}_i,\dots,w_k)\right)\\
&\qquad + \sum_{1\le i<j\le k}(-1)^{i+j}(f_s^*\iota_{X_s}\omega)_x\,([w_i,w_j],w_1,\dots,\hat{w}_i,\dots,w_k)\\
&\quad= (\mathrm{d}f_s^*\iota_{X_s}\omega)_x(w_1,\dots,w_k).
\end{aligned}
$$

Finally, putting everything together and using commutativity of exterior derivative with pullbacks, we get:

$$\frac{\mathrm{d}}{\mathrm{d}s} f_s^* \omega = (T_s \mathrm{d} + \mathrm{d} T_s) f^* \omega = f_s^*(\iota_{X_s} \mathrm{d}\omega) + \mathrm{d} f_s^* \iota_{X_s} \omega = f_s^*(\iota_{X_s} \mathrm{d}\omega + \mathrm{d}\iota_{X_s}\omega).$$

This completes the proof.

(e) Let $X \in \Gamma(TM)$ be a smooth vector field on M and let ψ_t be its flow. The Lie derivative $\mathcal{L}_X \omega \in \Omega^k(M)$ is defined by

$$\mathcal{L}_X \omega = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \psi_t^* \omega.$$

Show that

$$\mathcal{L}_X \omega = \iota_X \mathrm{d}\omega + \mathrm{d}\iota_X \omega.$$

Solution. This follows directly from Cartan's formula for $f_s = \psi_s$, $X_s = X$ and $s = 0$.