The most important exercises are marked with an asterisk *.

***3.1.**

(a) Find a symplectic manifold and a symplectomorphism on M that **is not** isotopic to the identity. (In particular, such a symplectomorphism is not a Hamiltonian diffeomorphism.)

Solution. Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. The standard symplectic form on $\mathbb{T}^2$ is the unique 2-form $\omega \in \Omega^2(\mathbb{T}^2)$ satisfying $\pi^*\omega = \omega_{\mathrm{std}}$ for the projection $\pi\colon \mathbb{R}^2 \to \mathbb{T}^2$. Let

$$\varphi\colon \mathbb{T}^2 \to \mathbb{T}^2, \qquad \varphi(x+\mathbb{Z}, y+\mathbb{Z}) = (-x+\mathbb{Z}, -y+\mathbb{Z}),$$

where $x, y \in \mathbb{R}$.

First note that the map φ is a symplectomorphism:

$$\begin{aligned}
\pi^*\varphi^*\omega &= (\varphi\circ\pi)^*\omega \\
&= \omega(\mathrm{d}(\varphi\circ\pi)(-), \mathrm{d}(\varphi\circ\pi)(-)) \\
&= \omega(-\mathrm{d}\pi(-), -\mathrm{d}\pi(-)) \\
&= \omega(\mathrm{d}\pi(-), \mathrm{d}\pi(-)) \\
&= \pi^*\omega = \omega_{\mathrm{std}}
\end{aligned}$$

hence $\varphi^*\omega = \omega$.

We claim that the map φ is not isotopic to the identity. Indeed, if it were, then the loops

$$x+\mathbb{Z} \mapsto \mathrm{id}(x+\mathbb{Z}, 0) = (x+\mathbb{Z}, 0) \qquad \text{and} \qquad x+\mathbb{Z} \mapsto \varphi(x+\mathbb{Z}, 0) = (-x+\mathbb{Z}, 0)$$

would be homotopic. Here we think of $x+\mathbb{Z}$ as as element of the circle $\mathbb{R}/\mathbb{Z}$. The second loop in exactly the reverse of the first loop and these loops are not homotopic in $\mathbb{T}^2$(see Algebraic Topology).

(b) Find a symplectic manifold and a symplectomorphism on M that **is** isotopic to the identity through symplectomorphisms, but is not a Hamiltonian diffeomorphism.

Hint: Consider translations on a cylinder.

Solution. Let $\Sigma = T^*S^1 = S^1 \times \mathbb{R}$ be a cylinder with coordinates (q, p) and let $\omega = dp \wedge dq = \mathrm{d}\lambda$, where $\lambda = p\mathrm{d}q$. We define

$$\varphi\colon [0,1] \times \Sigma \to \Sigma, \qquad \varphi_t(q, p) = (q, p+t).$$

We claim that φ_t is a symplectic isotopy. Indeed,

$$(\varphi_t^*\lambda)_{(q,p)}(\xi,\eta) = \lambda_{(q,p+t)}(\xi,\eta) = (p+t)\xi,$$

hence $\varphi_t^*\lambda_{(q,p)} = (p+t)\mathrm{d}q$. Thus

$$\varphi_t^*\omega = \mathrm{d}\varphi^*\lambda = \mathrm{d}(p+t)\wedge \mathrm{d}q = \mathrm{d}p\wedge\mathrm{d}q = \omega.$$

We now have to show that $\varphi = \varphi_1$ is not a Hamiltonian diffeomorphism. We do so by showing that all Hamiltonian diffeomorphisms have to satisfy equation (1) below, whereas φ does not.

Let ψ_t^H be a Hamiltonian diffeomorphism generated by a Hamiltonian $H \in C^\infty([0,1]\times\Sigma)$. Let $j\colon S^1 \to S^1\times\mathbb{R}, j(z) = (z,0)$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{S^1} j^*\left(\psi_t^H\right)^*\lambda &= \int_{S^1} j^*\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^H\right)^*\lambda\\ &= \int_{S^1} j^*\left(\psi_t^H\right)^*\left(-\mathrm{d}H_t + \mathrm{d}\iota_{X_t^H}\lambda\right)\\ &= \int_{S^1} \mathrm{d}j^*\left(\psi_t^H\right)^*\left(-H_t + \iota_{X_t^H}\lambda\right)\\ &= 0.\end{aligned}$$

It follows that

$$\int_{S^1} j^*\left(\psi_1^H\right)^*\lambda = \int_{S^1} j^*\left(\psi_0^H\right)^*\lambda = \int_{S^1} j^*\lambda = 0. \tag{1}$$

However,

$$\int_{S^1} j^*\varphi^*\lambda = \int_{S^1}\mathrm{d}q = 1 \neq 0.$$

This completes the proof.

(c) Does there exist a non-Hamiltonian symplectomorphism on S^2 equipped with the standard symplectic form, that is isotopic to the identity through symplectomorphisms?

Solution. No. Suppose φ_t is a symplectic isotopy. Then $\mathrm{d}\iota_{X_t}\omega = 0$, as we've seen in the lecture. Since $H^1(S^2;\mathbb{R}) = 0$, it follows that all closed 1-forms (in particular $\iota_{X_t}\omega$) are also exact. As shown in the lecture, there exists a smooth function $H_t\colon S^2\to\mathbb{R}$, smoothly depending on t, such that $\iota_{X_t}\omega = \mathrm{d}H_t$. This shows that φ_t is a Hamiltonian isotopy.

3.2. This exercise covers two useful facts from differential geometry and algebraic topology.

(a) Let $\omega_t \in \Omega^k(M)$ be a differential k-form and φ_t a smooth isotopy of diffeomorphisms. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\omega_t = \varphi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{\mathrm{d}}{\mathrm{d}t}\omega_t\right),$$

where X_t is the vector field defined by

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t = X_t \circ \varphi_t.$$

Solution. If $f(x,y)$ is a real function of two variables, then

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t,t) = \frac{\mathrm{d}}{\mathrm{d}x}f(x,t)\Big|_{x=t} + \frac{\mathrm{d}}{\mathrm{d}y}f(t,y)\Big|_{y=t}.$$

Therefore, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\omega_t &= \frac{\mathrm{d}}{\mathrm{d}x}\varphi_x^*\omega_t\Big|_{x=t} + \frac{\mathrm{d}}{\mathrm{d}y}\varphi_t^*\omega_y\Big|_{y=t}\\ &= \varphi_x^*\mathcal{L}_{X_x}\omega_t\Big|_{x=t} + \varphi_t^*\left(\frac{\mathrm{d}}{\mathrm{d}y}\omega_y\Big|_{y=t}\right)\\ &= \varphi_t^*\left(\mathcal{L}_{X_t}\omega_t + \frac{\mathrm{d}}{\mathrm{d}t}\omega_t\right).\end{aligned}$$

(b) Let $d \geq 1$ and $\alpha \in \Omega^d(\mathbb{R}^n)$ be a closed d-form, i.e. $\mathrm{d}\alpha = 0$. Show that α is exact, i.e. there exists $\lambda \in \Omega^{d-1}(\mathbb{R}^n)$ such that $\alpha = \mathrm{d}\lambda$. In other words, $\mathrm{H}^d(\mathbb{R}^n;\mathbb{R}) = 0$.

Hint: Use the retraction $f_t(x) = tx$ and the strategy we used in the lecture to show that "strongly isotopic" implies "isotopic".

Solution. For $t \in [0,1]$ consider the map $f_t(x) = tx$ on $\mathbb{R}^n$. This is a diffeomorphism for $t \neq 0$. We have $f_0 \equiv 0$ and $f_1 = \mathrm{id}$. For $t \in (0,1]$, let X_t be the smooth vector field associated to f_t, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t}f_t = X_t \circ f_t.$$

Then, as we show below, for all $0 < t_0 < t_1 < 1$, we have

$$f_{t_1}^*\alpha - f_{t_0}^*\alpha = \mathrm{d}Q\alpha + Q\mathrm{d}\alpha$$

where

$$Q\colon \Omega^{d-1}(\mathbb{R}^n) \to \Omega^d(\mathbb{R}^n), \qquad Q\alpha = \int_{t_0}^{t_1} f_t^*(\iota_{X_t}\alpha)\mathrm{d}t.$$

Indeed,

$$\begin{aligned}
\mathrm{d}Q\alpha + Q\mathrm{d}\alpha &= \mathrm{d}\int_{t_0}^{t_1} f_t^*(\iota_{X_t}\alpha)\mathrm{d}t + \int_{t_0}^{t_1} f_t^*(\iota_{X_t}(\mathrm{d}\alpha))\mathrm{d}t \\
&= \int_{t_0}^{t_1} f_t^*\left(\mathrm{d}\iota_{X_t}\alpha + \iota_{X_t}(\mathrm{d}\alpha)\right)\mathrm{d}t \\
&= \int_{t_0}^{t_1} f_t^*\left(\mathcal{L}_{X_t}\alpha\right)\mathrm{d}t \\
&\overset{(\star)}{=} \int_{t_0}^{t_1} \frac{\mathrm{d}}{\mathrm{d}t} f_t^*\alpha\,\mathrm{d}t \\
&= f_{t_1}^*\alpha - f_{t_0}^*\alpha.
\end{aligned}$$

In equality $(\star)$, we used Cartan's formula from Exercise 2.4; in the last equality, we used the Fundamental Theorem of Analysis. Using closedness of α and taking limits $t_0 \to 0, t_1 \to 1$, we conclude that

$$\alpha = f_1^*\alpha - f_0^*\alpha = \mathrm{d}\int_0^1 f_t^*(\iota_{X_t}\alpha)\mathrm{d}t.$$

is exact.

***3.3.** In this exercise, we prove Moser stability for volume forms. Let M be a closed smooth manifold of dimension m.

(a) Suppose $\mu_t \in \Omega^m(M)$, $t \in [0,1]$, is a smooth family of volume forms on M such that

(i) μ_t is a volume form for each t,

(ii) $\frac{\mathrm{d}}{\mathrm{d}t}\mu_t$ is exact for all $t \in [0,1]$.

Prove that there exists a smooth isotopy $\varphi_t\colon M \to M$ of diffeomorphisms on M satisfying $\varphi_t^*\mu_t = \mu_0$ for all $t \in [0,1]$.

Solution. Let $\beta_t \in \Omega^{n-1}(M)$ be an $(n-1)$-form depending smoothly on t and satisfying

$$\dot{\mu_t} = -\mathrm{d}\beta_t.$$

The fact that β_t can chosen to depend smoothly on t is again non-trivial, similarly to what we've seen in Moser stability of symplectic forms.

The assumption that all μ_t are volume forms allows us to define a vector field X_t by

$$\iota_{X_t}\mu_t = \beta_t.$$

Let φ_t be its flow (which exists because M is closed). Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t^*\mu_t &= \varphi_t^*\left(\mathcal{L}_{X_t}\mu_t + \dot{\mu}_t\right)\\ &= \varphi_t^*\left(\iota_{X_t}\mathrm{d}\mu_t + \mathrm{d}\iota_{X_t}\mu_t - \mathrm{d}\beta_t\right)\\ &= 0.\end{aligned}$$

Hence $\varphi_t^*\mu_t = \mu_0$ for all t.

(b) Let $\mu_0, \mu_1 \in \Omega^m(M)$ be two volume forms on M such that

$$\int_M \mu_0 = \int_M \mu_1.$$

Prove that there exists a diffeomorphism $\varphi\colon M \to M$, isotopic to id, satisfying $\varphi^*\mu_1 = \mu_0$.

Solution. Consider $\mu_t = (1-t)\mu_0 + t\mu_1$. Then μ_t is a volume form for all t and $[\mu_t]$ is constant. Therefore part (a) applies.

3.4. Let (Σ, σ) and (Σ', σ') be two closed connected symplectic surfaces. Suppose Σ has total area 1 and Σ' has total area c. Let $a \in \mathbb{R}\backslash 0$. Endow the product manifold $\Sigma \times \Sigma'$ with the symplectic form $\omega_a = a\sigma \oplus a^{-1}\sigma'$.

(a) Show that (M, ω_a) all have the same volume.

Solution. By definition, $\omega_a = a\sigma \oplus a^{-1}\sigma' = \pi^*\sigma + \pi'^*\sigma'$, where $\pi\colon \Sigma \oplus \Sigma' \to \Sigma$ and $\pi'\colon \Sigma \oplus \Sigma' \to \Sigma'$ are the projections. Hence

$$\begin{aligned}\omega_a^{\wedge 2} &= \left(\pi^*(a\sigma) + \pi'^*(a^{-1}\sigma')\right)^{\wedge 2}\\ &= \pi^*(a^2\sigma^{\wedge 2}) + 2\pi^*(a\sigma) \wedge \pi'^*(a^{-1}\sigma') + \pi^*(a^{-2}\sigma'^{\wedge 2})\\ &= 2\pi^*\sigma \wedge \pi'^*\sigma' = \omega_1^{\wedge 2}\end{aligned}$$

is independent of a. In particular,

$$\mathrm{vol}(M, \omega_a) = \frac{1}{2!}\int_M \omega_a^{\wedge 2} = \frac{1}{2!}\int_M \omega_1^{\wedge 2} = \mathrm{vol}(M, \omega_1)$$

(b) Show that there exist a such that (M, ω_1) and (M, ω_a) are not symplectomorphic.

Hint: The Degree Theorem from Algebraic Topology tells us the following. Let X and Y be compact oriented manifolds of same dimension and let $f\colon X \to Y$ be a smooth map. Then every top degree form Ω satisfies

$$\int_X f^*\Omega = \deg f \int_Y \Omega$$

Since in Exercise 3.4 (a) we saw that all (M, ω_a) have the same volume, try instead using the Degree Theorem to compare volumes of the projections on Σ of ω_a calculated directly and as $\varphi^*\omega_1$ for φ a symplectomorphism.

Solution. Suppose (M, ω_1) and (M, ω_a) are symplectomorphic. Choose a diffeomorphism $\varphi\colon M \to M$ such that $\omega_a = \varphi^*\omega_1$. Fix $z_0' \in \Sigma'$ and consider $j\colon \Sigma \to M, z \mapsto (z, z_0')$.

Note that if $\omega_a = \varphi^*\omega_1$, then also $j^*\omega_a = j^*\varphi^*\omega_1$ and in particular it must hold that:

$$\int_\Sigma j^*\omega_a = \int_\Sigma j^*\varphi^*\omega_1. \tag{2}$$

We now calculate the integrals. For the first one, we first note

$$\begin{aligned} j^*\omega_a &= j^*\left(\pi^*(a\sigma) + \pi'^*(a^{-1}\sigma')\right) \\ &= (\pi \circ j)^*(a\sigma) + (\pi' \circ j)^*(a^{-1}\sigma) \\ &= \mathrm{id}^*(a\sigma) + (c_{z_0'})^*(a^{-1}\sigma) \\ &= a\sigma \end{aligned}$$

and hence

$$\int_\Sigma j^*\omega_a = \int_\Sigma a\sigma = a\int_\Sigma \sigma = a.$$

On the other hand, using the Degree Theorem, for the second integral we get:

$$\begin{aligned} \int_\Sigma j^*\varphi^*\omega_1 &= \int_\Sigma j^*\varphi^*(\sigma \oplus \sigma') \\ &= \int_\Sigma (\pi \circ \varphi \circ j)^*\sigma + (\pi' \circ \varphi \circ j)^*\sigma' \\ &= \deg(\pi \circ \varphi \circ j)\int_X \sigma + \deg(\pi' \circ \varphi \circ j)\int_{X'} \sigma' \\ &= \deg(\pi \circ \varphi \circ j) + \deg(\pi' \circ \varphi \circ j)c \in \mathbb{Z} + c\mathbb{Z}. \end{aligned}$$

Substituting these back into equation (2), we see that $a \in \mathbb{Z} + c\mathbb{Z}$. In particular, for $a \notin \mathbb{Z} + c\mathbb{Z}$ the two symplectic manifolds (M, ω_1) and (M, ω_a) are not symplectomorphic.