The most important exercises are marked with an asterisk *.

***4.1.** Let $\mathrm{Sp}(2n)$ be the group of symplectic matrices

$$\mathrm{Sp}(2n) = \left\{ A \in \mathrm{GL}(2n, \mathbb{R}) \mid A^T J_0 A = J_0 \right\},$$

where

$$J_0 = \begin{pmatrix} 0 & \mathrm{id}_n \\ -\mathrm{id}_n & 0 \end{pmatrix}.$$

(a) Show that if $\Psi \in \mathrm{Sp}(2n)$ then $\Psi^{-1} \in \mathrm{Sp}(2n)$ and $\Psi^T \in \mathrm{Sp}(2n)$.

Solution. First we show that Ψ^{-1} is symplectic:

$$\Psi^{-T} J_0 \Psi^{-1} = \Psi^{-T}(\Psi^T J_0 \Psi)\Psi^{-1} = J_0 \Longrightarrow \Psi^{-1} \in \mathrm{Sp}(2n).$$

Taking the inverse of $\Psi^T J_0 \Psi = J_0$ we get

$$\begin{aligned} -\Psi^{-1} J_0 \Psi^{-T} &= \Psi^{-1} J_0^{-1} \Psi^{-T} = J_0^{-1} = -J_0 \\ &\Longrightarrow \Psi^{-1} J_0 \Psi^{-T} = J_0 \\ &\Longrightarrow J_0 = \Psi J_0 \Psi^T = (\Psi^T)^T J_0 \Psi^T \\ &\Longrightarrow \Psi^T \in \mathrm{Sp}(2n). \end{aligned}$$

(b) Show that if $P \in \mathrm{Sp}(2n)$ is a symmetric, positive definite symplectic matrix, then $P^\alpha \in \mathrm{Sp}(2n)$ for every $\alpha \geq 0$, $\alpha \in \mathbb{R}$.

Solution. To see that we can define a power of P we decompose $\mathbb{R}^{2n}$ into the eigenspaces of P:

$$\mathbb{R}^{2n} = \oplus_{i=1}^k E_i, \; E_i = \ker(\lambda_i \,\mathrm{id} - P),$$

where $\lambda_1, \dots, \lambda_k > 0$ are the eigenvalues of P. Let $\alpha \geq 0$. The power P^α is defined by $P^\alpha(v) = \lambda_i^p v$ for $v \in E_i$. Let $v \in E_i$ and $w \in E_j$. Then, since P is a symplectic matrix,

$$\omega_{\mathrm{std}}(v, w) = \omega_{\mathrm{std}}(Pv, Pw) = \lambda_i \lambda_j \omega_{\mathrm{std}}(v, w).$$

In particular, either $\lambda_i \lambda_j = 1$ or $\omega_{\mathrm{std}}(v, w) = 0$. This implies

$$\omega_{\mathrm{std}}(P^\alpha v, P^\alpha w) = \omega_{\mathrm{std}}(\lambda_i^\alpha v, \lambda_j^\alpha w) = (\lambda_i \lambda_j)^\alpha \omega_{\mathrm{std}}(v, w) = \omega_{\mathrm{std}}(v, w).$$

Therefore, P^α is symplectic.

(c) Show that $\mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{U}(n)$ and that the inclusion $\mathrm{U}(n) \subset \mathrm{Sp}(2n)$ is a homotopy equivalence.

Hint: Consider the homotopy $f_t(\Psi) = \Psi \left(\Psi^T \Psi\right)^{-\frac{t}{2}}$, $t \in [0,1]$.

Solution. A direct calculation shows that the subgroup $\mathrm{Sp}(2n) \cap O(2n)$ consists of those matrices

$$\Psi = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \in \mathrm{GL}(2n, \mathbb{R})$$

which satisfy $X^T Y = Y^T X$ and $X^T X + Y^T Y = \mathrm{id}$. This is precisely the condition for the complex matrix $U := X + iY$ to be unitary. This shows $\mathrm{Sp}(2n) \cap \mathrm{O}(2n) = \mathrm{U}(n)$.

Let $\Psi \in \mathrm{Sp}(2n)$. By part (a), $P := \Psi^T \Psi$ is a symplectic matrix. Moreover, P is symmetric and positive definite. Therefore we can apply part (b) to P and thus

$$\left(\Psi^T \Psi\right)^{-\frac{t}{2}} \in \mathrm{Sp}(2n).$$

In particular, $f_t \colon \mathrm{Sp}(2n) \to \mathrm{Sp}(2n)$ defined as in the hint is well-defined and takes values in $\mathrm{Sp}(2n)$.

We claim that $r \colon \mathrm{Sp}(2n) \to \mathrm{U}(n)$, $r(\Psi) = f_1(\Psi)$ is a homotopy inverse to the inclusion $i \colon \mathrm{U}(n) \hookrightarrow \mathrm{Sp}(2n)$. Indeed, $f_1(\Psi) \in \mathrm{Sp}(2n) \cap \mathrm{O}(n) = \mathrm{U}(n)$ because

$$\begin{aligned} f_1(\Psi) f_1(\Psi)^T &= \Psi \left(\Psi^T \Psi\right)^{-\frac{1}{2}} \left(\left(\Psi^T \Psi\right)^{-\frac{1}{2}}\right)^T \Psi^T \\ &= \Psi \left(\Psi^T \Psi\right)^{-1} \Psi^T \\ &= \Psi \Psi^{-1} \Psi^{-T} \Psi^T \\ &= \mathrm{id}, \end{aligned}$$

hence r is well-defined. Moreover, f_t is a homotopy from $f_0 = \mathrm{id}$ to $f_1 = i \circ r$. For the converse composition, $r \circ i = \mathrm{id}$ because if $\Psi \in \mathrm{U}(n)$, then $\Psi^T \Psi = \mathrm{id}$ and thus $r(\Psi) = f_1(\Psi) = \Psi$.

4.2.

(a) Let $\Omega(\mathbb{R}^{2n})$ denote the space of linear symplectic forms on $\mathbb{R}^{2n}$. Show that

$$\Omega(\mathbb{R}^{2n}) \cong \mathrm{GL}(2n, \mathbb{R}) / \mathrm{Sp}(2n).$$

Solution. Consider the map

$$\Psi\colon\ \mathrm{GL}(2n,\mathbb{R}) \to \Omega(\mathbb{R}^{2n}),\ \Psi(A) := A^*\omega_{\mathrm{std}}.$$

We've seen in the lecture that this map is surjective. The kernel of Ψ is precisely the group $\mathrm{Sp}(2n)$. The claimed isomorphism follows.

(b) Deduce that $\Omega(\mathbb{R}^{2n})$ is homotopy equivalent to $\mathrm{O}(2n)/\,\mathrm{U}(n)$.

Hint: Use a similar strategy as in Exercise 4.1 (c).

Solution. Consider the map

$$j\colon\ \mathrm{O}(2n)/\,\mathrm{U}(2n) \to \mathrm{GL}(2n,\mathbb{R})/\,\mathrm{Sp}(2n)$$

induced by the inclusion $\mathrm{O}(2n) \hookrightarrow \mathrm{GL}(2n,\mathbb{R})$. We show that this is a homotopy equivalence. As in Exercise 4.1, we consider the homotopy

$$f_t(\Psi) = \Psi\left(\Psi^T\Psi\right)^{-\frac{t}{2}},\ t \in [0,1].$$

This time we view it as an automorphism on $\mathrm{GL}(2n,\mathbb{R})$. Since $f_t(\mathrm{Sp}(2n)) \subset \mathrm{Sp}(2n)$ as shown in Exercise 4.1, the map f_t descends to a map

$$F_t\colon\ \mathrm{GL}(2n,\mathbb{R})/\,\mathrm{Sp}(2n) \to \mathrm{GL}(2n,\mathbb{R})/\,\mathrm{Sp}(2n),\ F_t([\Psi]) = [f_t(\Psi)].$$

Moreover, $f_1(\Psi) \in \mathrm{O}(2n)$ and $f_1(\mathrm{Sp}(2n)) \subset \mathrm{U}(n)$ and therefore F_1 defines a map $R\colon\ \mathrm{GL}(2n,\mathbb{R})/\,\mathrm{Sp}(2n) \to \mathrm{O}(2n)/\,\mathrm{U}(n)$, $R([\Psi]) = [f_1(\Psi)]$.

The map F_t is a homotopy from $F_0 = \mathrm{id}$ to $F_1 = j \circ R$. Finally $R \circ j = \mathrm{id}$ and hence R is a homotopy inverse to j.

(c) Let $\mathcal{J}(\mathbb{R}^{2n})$ denote the space of linear complex structures on $\mathbb{R}^{2n}$. Show that

$$\mathcal{J}(\mathbb{R}^{2n}) \cong \mathrm{GL}(2n,\mathbb{R})/\,\mathrm{GL}(n,\mathbb{C}).$$

Solution. Consider the map

$$\Phi\colon\ \mathrm{GL}(2n,\mathbb{R}) \to \mathcal{J}(\mathbb{R}^{2n}),\ \Phi(A) := AJ_0A^{-1}.$$

We've seen in the lecture that this map is surjective. The kernel of Φ is precisely the group $\mathrm{GL}(n,\mathbb{C})$. The claimed isomorphism follows.

(d) Show that $\mathrm{GL}(n,\mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{U}(n)$ and that $\mathcal{J}(\mathbb{R}^{2n})$ is homotopy equivalent to $\mathrm{O}(2n)/\,\mathrm{U}(n)$. In particular, $\Omega(\mathbb{R}^{2n})$ and $\mathcal{J}(\mathbb{R}^{2n})$ are homotopy equivalent.

Solution. Let $\Psi \in \mathrm{GL}(2n,\mathbb{R})$. Then

$$\Psi \in \mathrm{GL}(n,\mathbb{C}) \Longrightarrow \Psi J_0 = J_0 \Psi.$$

If $\Psi \in \mathrm{GL}(n,\mathbb{C}) \cap \mathrm{O}(2n)$, then

$$\Psi^T J_0 \Psi = \Psi^T \Psi J_0 = J_0.$$

It follows that $\Psi \in \mathrm{Sp}(2n)$. If $\Psi \in \mathrm{Sp}(2n) \cap \mathrm{O}(2n)$, then

$$\Psi J_0 = \Psi(\Psi^T J_0 \Psi) = J_0 \Psi,$$

hence $\Psi \in \mathrm{GL}(n,\mathbb{C})$. Putting these two observations together shows

$$\mathrm{GL}(n,\mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{Sp}(2n) \cap \mathrm{O}(2n)$$

and the first claim follows from Exercise 4.1 (c).

We now show that $\mathrm{GL}(2n,\mathbb{R})/\,\mathrm{GL}(n,\mathbb{C})$ is homotopy equivalent to $\mathrm{O}(2n)/\,\mathrm{U}(n)$. For this, consider the map

$$k\colon\ \mathrm{O}(2n)/\,\mathrm{U}(n) \to \mathrm{GL}(2n,\mathbb{R})/\,\mathrm{GL}(n,\mathbb{C})$$

induced by inclusion. We will again make use of the homotopy f_t defined in the hint to Exercise 4.1 (c). As a preparation, we show a complex analogue of Exercise 4.1 (b):

Claim 1. *Let $P \in \mathrm{GL}(n,\mathbb{C})$ be a symmetric positive definite matrix and $\alpha \geq 0$. Then $P^\alpha \in \mathrm{GL}(n,\mathbb{C})$.*

Proof. Let E_i be the eigenspaces of P and $\lambda_i > 0$ the eigenvalues as in Exercise 4.1.(b). Since J_0 commutes with P, it follows that J_0 preserves the eigenspaces. Let $v \in E_i$. Then

$$P^\alpha(J_0(z)) = \lambda_i^\alpha(J_0(z)) = J_0(\lambda_i^\alpha z) = J_0(P^\alpha(z)).$$

Therefore, $P^\alpha J_0 = J_0 P^\alpha$ and thus $P^\alpha \in \mathrm{GL}(n,\mathbb{C})$. □

Let $\Psi \in \mathrm{GL}(n,\mathbb{C})$. We apply the claim to the symmetric positive definite matrix $P = \Psi^T \Psi$. It follows that $f_t(\Psi) \in \mathrm{GL}(n,\mathbb{C})$. In particular, f_t induces a map

$$\tilde{F}_t\colon\ \mathrm{GL}(2n,\mathbb{R})/\,\mathrm{GL}(n,\mathbb{C}) \to \mathrm{GL}(2n,\mathbb{R})/\,\mathrm{GL}(n,\mathbb{C}),\ \tilde{F}_t([\Psi]) = [f_t(\Psi)].$$

Moreover, $f_1(\Psi) \in \mathrm{O}(2n)$ and $f_1(\mathrm{GL}(n,\mathbb{C})) \subset \mathrm{GL}(n,\mathbb{C}) \cap \mathrm{O}(2n) = \mathrm{U}(n)$. Therefore $\tilde{F}_1$ defines a map $\tilde{R}\colon \mathrm{GL}(2n,\mathbb{R})/\mathrm{GL}(n,\mathbb{C}) \to \mathrm{O}(2n)/\mathrm{U}(n)$, $\tilde{R}([\Psi]) = [f_1(\Psi)]$.

The map $\tilde{F}_t$ is a homotopy from $\tilde{F}_0 = \mathrm{id}$ to $\tilde{F}_1 = k \circ \tilde{R}$. Finally $\tilde{R} \circ k = \mathrm{id}$ and hence $\tilde{R}$ is a homotopy inverse to k.

***4.3.**

(a) Show that any co-oriented hypersurface $\Sigma \subset \mathbb{R}^3$ inherits an almost complex structure from the vector product as follows. Let $\nu\colon \Sigma \to S^2$ be the Gauss map. Then

$$J_x(u) := \nu(x) \times u$$

is an almost complex structure.

Solution. We compute J_x^2:

$$\begin{aligned} J_x(J_x(u)) &= J_x(\nu(x) \times u) \\ &= \nu(x) \times (\nu(x) \times u) \\ &= \nu(x)(u \cdot \nu(x)) - u(\nu(x) \cdot \nu(x) \\ &= -u, \end{aligned}$$

where the last identity follows from the formula

$$a \times (b \times c) = b(c \cdot a) - c(a \cdot b).$$

and the fact that $\nu(x)$ is a unit normal vector (i.e. $u \cdot \nu(x) = 0$ and $\nu(x) \cdot \nu(x) = 1$). It follows that $J_x^2 = -\mathrm{id}$ and J_x is a linear complex structure for each x. Hence J is an almost complex structure on Σ.

(b) Show that J is compatible with the symplectic form

$$\omega_x(v, w) = \nu(x) \cdot (v \times w)$$

(see Exercise 1.3).

Solution. For $v, w \in T_x\Sigma$ we compute

$$\begin{aligned} \omega_x(v, J_x w) &= \nu(x) \cdot (v \times (\nu(x) \times w)) \\ &= \nu(x) \cdot (\nu(x)(w \cdot v) - w(v \cdot \nu(x))) \\ &= w \cdot v. \end{aligned}$$

Hence $\omega_x(-, J_x-)$ is the standard scalar product on $T_x\Sigma$. In partiucular, J and ω are compatible.

(c) Show that every co-oriented hypersurface $M \subset \mathbb{R}^7$ also carries an almost complex structure.

Solution. Since $\mathbb{R}^7$ also carries a vector product, the exact same proof as in (a) shows that any co-oriented hypersurface $M \subset \mathbb{R}^7$ carries an almost complex structure.

(d) Give an example of an almost complex manifold that does not admit a symplectic structure.

Solution. The sphere $S^6 \subset \mathbb{R}^7$ admits an almost complex structure by part (c). However, since $H^2(S^6) = 0$, it does not admit a symplectic structure.

Remark: There is a non-degenerate 2-form ω on S^6 given by the same formula as in (b). J is also compatible with this 2-form. However, ω is not closed.