The most important exercises are marked with an asterisk *.

5.1. Let $\omega \in \Omega^2(M)$ be a non-degenerate 2-form and $J \in \mathcal{J}_c(M,\omega)$. Let ∇ denote the Levi-Civita connection associated to the Riemannian metric $g_J(v,w) := \omega(v, Jw)$.

(a) Show that for any $X \in \Gamma(TM)$, we have

$$(\nabla_X J)J + J(\nabla_X J) = 0.$$

Solution. Since the map J can be thought of as a section of the Hom-bundle $\mathrm{Hom}(TM,TM) \to M$, the connection applied to J is given by

$$(\nabla_X J)Y = \nabla_X(JY) - J\nabla_X Y.$$

We thus have:

$$\begin{aligned}(\nabla_X J)JY + J((\nabla_X J)Y) &= \nabla_X(J^2Y) - J\nabla_X(JY) + J\nabla_X(JY) - J^2\nabla_X Y\\ &= \nabla_X(-Y) - (-\nabla_X Y)\\ &= -\nabla_X Y + \nabla_X Y = 0.\end{aligned}$$

(b) Let $X, Y, Z \in \Gamma(TM)$ be three vector fields. Show that

$$g_J((\nabla_X J)Y, Z) + g_J(Y, (\nabla_X J)Z) = 0.$$

Solution. Recall that, since ∇ is a Levi-Civita connection, it satisfies the Ricci identity:

$$X(g(Y,Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z).$$

Note that

$$g_J(JY,Z) + g_J(Y,JZ) = \omega(JY,JZ) + \omega(Y,-Z) = 0.$$

Differentiating the above identity in direction of X, we get

$$\begin{aligned}0 &= X(g_J(JY,Z) + g_J(Y,JZ))\\ &= g_J(\nabla_X(JY),Z) + g_J(JY,\nabla_X Z) + g_J(\nabla_X Y, JZ) + g_J(Y,\nabla_X(JZ))\\ &= g_J((\nabla_X J)Y,Z) + g_J(J\nabla_X Y, Z) + g_J(JY,\nabla_X Z) + g_J(\nabla_X Y, JZ)\\ &\quad + g_J(Y,(\nabla_X J)Z) + g_J(Y, J\nabla_X Z)\\ &= g_J((\nabla_X J)Y,Z) + g_J(J\nabla_X Y, Z) + g_J(JY,\nabla_X Z) - g_J(J\nabla_X Y, Z)\\ &\quad + g_J(Y,(\nabla_X J)Z) - g_J(JY,\nabla_X Z)\\ &= g_J((\nabla_X J)Y,Z) + g_J(Y,(\nabla_X J)Z)\end{aligned}$$

(c) Show that

$$d\omega = g_J((\nabla_X J)Y, Z) + g_J((\nabla_Y J)Z, X) + g_J((\nabla_Z J)X, Y).$$

Solution. We use the identity

$$\begin{aligned} d\omega(X,Y,Z) = &\nabla_X(\omega(Y,Z)) + \nabla_Y(\omega(Z,X)) + \nabla_Z(\omega(X,Y)) \\ &- \omega([X,Y],Z) - \omega([Y,Z],X) - \omega([Z,X],Y), \end{aligned}$$

where $[X,Y] = \nabla_X Y - \nabla_Y X$. Plugging in $\omega(v,w) = g_J(Jv,w)$ we get

$$\begin{aligned} \nabla_X(\omega(Y,Z)) - \omega([X,Y],Z) &= \nabla_X(g_J(JY,Z)) - g_J(J(\nabla_X Y - \nabla_Y X), Z) \\ &= g_J(\nabla_X(JY), Z) + g_J(JY, \nabla_X Z) \\ &\quad - g_J(J\nabla_X Y, Z) + g_J(J\nabla_Y X, Z) \\ &= g_J((\nabla_X J)Y, Z) + g_J(JY, \nabla_X Z) + g_J(J\nabla_Y X, Z). \end{aligned}$$

Similarly we treat the other terms in (c). We therefore get

$$\begin{aligned} d\omega(X,Y,Z) &= g_J((\nabla_X J)Y,Z) + g_J(JY,\nabla_X Z) + g_J(J\nabla_Y X, Z) \\ &\quad g_J((\nabla_Y J)Z,X) + g_J(JZ,\nabla_Y X) + g_J(J\nabla_Z Y, X) \\ &\quad g_J((\nabla_Z J)X,Y) + g_J(JX,\nabla_Z Y) + g_J(J\nabla_X Z, Y) \\ &= g_J((\nabla_X J)Y,Z) + g_J(JY,\nabla_X Z) - g_J(\nabla_Y X, JZ) \\ &\quad g_J((\nabla_Y J)Z,X) + g_J(JZ,\nabla_Y X) - g_J(\nabla_Z Y, JX) \\ &\quad g_J((\nabla_Z J)X,Y) + g_J(JX,\nabla_Z Y) - g_J(\nabla_X Z, JY) \\ &= g_J((\nabla_X J)Y,Z) + g_J((\nabla_Y J)Z,X) + g_J((\nabla_Z J)X,Y). \end{aligned}$$

5.2. Let $\omega \in \Omega^2(M)$, $J \in \mathcal{J}_c(M,\omega)$, g_J and ∇ be as above. Show that the following are equivalent:

(i) $\nabla J = 0$

(ii) J is integrable and ω is closed.

Solution. If $\nabla J = 0$, then $d\omega = 0$ follows from 5.1.(c). To show that J is integrable, we expand the Nijenhuis tensor:

$$\begin{aligned} N_J(X,Y) &= [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y] \\ &= \nabla_{JX}(JY) - \nabla_{JY}(JX) - J\nabla_{JX}Y + J\nabla_Y(JX) \\ &\quad - J\nabla_X(JY) + J\nabla_{JY}X - \nabla_X Y + \nabla_Y X \\ &= (\nabla_{JX}J)Y + J\nabla_{JX}Y - (\nabla_{JY}J)X - J\nabla_{JY}X \\ &\quad - J\nabla_{JX}Y + J(\nabla_Y J)X + J^2\nabla_Y X \\ &\quad - J(\nabla_X J)Y - J^2\nabla_X Y + J\nabla_{JY}X - \nabla_X Y + \nabla_Y X \\ &= (\nabla_{JX}J)Y - (\nabla_{JY}J)X + (\nabla_X J)JY - (\nabla_Y J)JX, \end{aligned}$$

where in the last equality we used 5.1.(a). Hence $N_J = 0$ if $\nabla J = 0$. We conclude that J is integrable by the Newlander–Nirenberg Theorem.

Conversely, suppose J is integrable and ω is closed. Using the above calculation we see that

$$\begin{aligned} g_J(N_J(X,Y),Z) &= g_J((\nabla_{JX}J)Y - (\nabla_{JY}J)X + (\nabla_X J)JY - (\nabla_Y J)JX, Z) \\ &\overset{5.1.(b)}{=} g_J((\nabla_{JX}J)Y,Z) + g_J(X,(\nabla_{JY}J)Z) + g_J(\nabla_X J)JY,Z) + g_J(JX,(\nabla_Y J)Z) \\ &\overset{5.1.(a)}{=} g_J((\nabla_{JX}J)Y,Z) + g_J((\nabla_Y J)Z,JX) + g_J((\nabla_Z J)JX,Y) \\ &\quad + g_J((\nabla_X J)JY,Z) + g_J((\nabla_{JY}J)Z,X) + g_J((\nabla_Z J)X,JY) \\ &\quad + 2g_J(J(\nabla_Z J)X,Y) \\ &\overset{5.1.(c)}{=} \mathrm{d}\omega(JX,Y,Z) + \mathrm{d}\omega(X,JY,Z) + 2g_J(J(\nabla_Z J)X,Y). \end{aligned}$$

We conclude that if $N_J = 0$ and $\mathrm{d}\omega = 0$, then the term $g_J(J(\nabla_Z J)X,Y)$ has to vanish for all X, Y, Z and thus $\nabla_Z J = 0$ for all Z.

***5.3.** Let $B(r) \subset \mathbb{R}^2$ denote the open disc of radius r. We use the coordinates x_1, y_1, x_2, y_2 and the symplectic form $\mathrm{d}y_1 \wedge \mathrm{d}x_1 + \mathrm{d}y_2 \wedge \mathrm{d}x_2$ on $\mathbb{R}^4$. Consider the product $B(r) \times B\left(\frac{1}{r}\right) \subset \mathbb{R}^4$.

(a) Show that there exists a volume preserving diffeomorphism

$$\psi\colon B(1) \times B(1) \to B(r) \times B\left(\frac{1}{r}\right)$$

for any $r > 0$.

Solution. The map

$$\psi(x_1,\dots,x_n,y_1,\dots,y_n) = (rx_1,\dots,rx_n,\frac{1}{r}y_1,\dots,\frac{1}{r}y_n)$$

is a volume-preserving map on $\mathbb{R}^{2n}$ restricting do a diffeomorphism from $B(1) \times B(1)$ to $B(r) \times B\left(\frac{1}{r}\right)$.

(b) Let c be symplectic capacity in dimension 4. Show that

$$c\left(B(r) \times B\left(\frac{1}{r}\right), \omega_{\mathrm{std}}\right) \to 0$$

as $r \to 0$.

Solution. For any $r > 0$ we can symplectically embed $B(r) \times B\left(\frac{1}{r}\right)$ into the cylinder $Z(r)$ by the standard inclusion. In particular,

$$0 < c\left(B(r) \times B\left(\frac{1}{r}\right)\right) \leq c(Z(r)) = \pi r^2.$$

So clearly $c\left(B(r) \times B\left(\frac{1}{r}\right)\right) \to 0$ as $r \to 0$.

(c) Let $0 < r_1 \leq r_2$ and $0 < s_1 \leq s_2$. Show that there exists a symplectic diffeomorphism

$$\varphi\colon B(r_1) \times B(r_2) \to B(s_1) \times B(s_2)$$

if and only if $r_1 = s_1$ and $r_2 = s_2$.

Hint: You may use the fact that a symplectic capacity exists.

Solution. Suppose φ is a symplectic diffeomorphism. Consider the symplectic embedding

$$B^4(r_1) \to B(r_1) \times B(r_2) \xrightarrow{\varphi} B(s_1) \times B(s_2) \to B(s_1) \times \mathbb{R}^2 = Z(s_1).$$

It follows from monotonicity of a symplectic capacity c that

$$\pi r_1^2 \leq c(B^4(r_1)) \leq c(Z(s_1)) = \pi s_1^2,$$

hence $r_1 \leq s_1$. Applying the same argument to φ^{-1} yields $s_1 \leq r_1$, thus $r_1 = s_1$. Now φ is also volume-preserving, hence $r_1 r_2 = s_1 s_2$. The result follows.

Remark: The generalization of (c) to the product of n open symplectic 2-balls in $\mathbb{R}^{2n}$ is true. The proof is more subtle and needs more machinery (e.g. symplectic homology).

***5.4.** Given a linear subspace $W \subset \mathbb{R}^{2n}$, its *symplectic complement* is defined by

$$W^\perp = \{v \in \mathbb{R}^{2n} \,|\, \omega_{\text{std}}(v, w) = 0 \text{ for all } w \in W\}.$$

The subspace W is called *isotropic* if $W \subset W^\perp$.

(a) Show that $(W^\perp)^\perp = W$ and $\dim W^\perp = \dim \mathbb{R}^{2n} - \dim W$.

Solution. Define the linear map

$$\iota_\omega\colon \mathbb{R}^{2n} \to W^*, w \mapsto \omega_{\text{std}}(-, w).$$

This has $W^\perp$ as kernel and it is surjective because ω_{std} is non-degenerate. Therefore

$$\dim \mathbb{R}^{2n} = \dim \ker \iota_\omega + \dim \operatorname{im} \iota_\omega = \dim W^\perp + \dim W^* = \dim W^\perp + \dim W.$$

$(W^\perp)^\perp = W$ is a direct calculation.

(b) Show that if W is isotropic then $\dim W \leq n$.

Solution. By the previous exercise we have

$$2n = \dim \mathbb{R}^{2n} = \dim W + \dim W^{\perp} \geq 2 \dim W,$$

where we used $\dim W \leq \dim W^{\perp}$ for isotropic W. Thus $n \geq \dim W$.

(c) Let c be a symplectic capacity. Let $\Omega \subset \mathbb{R}^{2n}$ be an open bounded set containing 0 and $W \subset \mathbb{R}^{2n}$ a linear subspace of codimension 2. Show that

$$c(\Omega + W) = +\infty$$

if $W^{\perp}$ is isotropic. Here,

$$\Omega + W = \{x + w \in \mathbb{R}^{2n} \mid x \in \Omega, w \in W\}.$$

Solution. Since $\operatorname{codim} W = 2$ it follows from part (a) that $\dim W^{\perp} = 2$. Since ω vanishes on $W^{\perp}$ we can assume that

$$W = \{(x, y) \mid x_1 = x_2 = 0\}.$$

Indeed, we claim that for any s-dimensional subspace $S \subset V$ of an $2n$-dimensional vector space V such that $\omega|_S = 0$ a basis $\{a_1, \dots, a_s\}$ of S can be extended to a symplectic basis of V. To prove the claim, we argue as follows. If $s = n$, we have a Lagrangian subspace and we saw this case in lectures. If $s < n$, then the system

$$\omega(a_1, v) = 0, \qquad \omega(a_2, v) = 0, \qquad \dots \qquad \omega(a_s, v) = 0 \qquad \text{for } v \in V$$

consists of s linearly independent equations. Since $\dim V = 2n$, the system has $2n - s$ independent solutions. There will thus exist some $a_{s+1} \in V$ that is linearly independent of $\{a_1, \dots, a_s\}$ and that satisfies the system above. This process can be repeated until we get a set $\{a_1, \dots, a_n\}$ of linearly independent vectors satisfying $\omega(a_i, a_j) = 0$ for $1 \leq i, j \leq n$.

Using non-degenerecy of ω we can find vectors c_i for $1 \leq i \leq n$ such that $\omega(a_i, c_j) = \delta_{ij}$ and such that the $2n$ vectors a_i and c_j are linearly independent. We are not yet done because $\omega(c_i, c_j) \neq 0$. To fix this, set $b_i = c_j + \sum_{j=1}^{n} s_{ij} a_j$. To get the constants s_{ij}, we solve $\omega(b^i, b^j) = 0$. This concludes the proof of our claim.

Now applying the claim to $V = \mathbb{R}^{2n}$ and $S = W^{\perp}$, we get a symplectic basis such that $W^{\perp} = \operatorname{span}\{a_1, a_2\}$. We can then apply linear symplectic transformation to send $a_1, \dots, a_s$ to $e_1, \dots, e_s$, where $\{e_1, \dots, e_n, f_1, \dots, f_n\}$ is the standard symplectic basis.

We show that any ball can be symplectically embedded into $\Omega + W$. To see this, define the linear symplectic map

$$\varphi(x, y) \mapsto \left(\epsilon x, \frac{1}{\epsilon} y\right).$$

For small enough ϵ, the map φ restricts to a symplectic embedding of $B(R) \to \Omega + W$. By monotonicity

$$\pi R^2 = c(B(R)) \leq c(\Omega + W).$$

Since $R > 0$ was arbitrary, it follows that $c(\Omega + W) = +\infty$.

(d) Let $\Omega \subset \mathbb{R}^{2n}$ and $W \subset \mathbb{R}^{2n}$ be as above. Show that

$$0 < c(\Omega + W) < +\infty$$

if $W^\perp$ is not isotropic.

Solution. Since $W^\perp$ has dimension 2 and is not isotropic, it is actually symplectic, i.e. $\omega|_{W^\perp}$ is non-degenerate. Indeed, let $u \in W^\perp$. We claim there exists $v \in W^\perp$ that is linearly independent from u and such that $\omega(u, v) \neq 0$. If such a v did not exist, then we would have $\omega(u, v) = 0$, which would imply $v \in W$ by the definition of $W^\perp$ and the fact that $u \in W^\perp$. But similarly, since $v \in W^\perp$, we would have that $\omega(v, u) = 0$, which implies $u \in W$. In other words, $W^\perp \subset W$. This contradicts our assumption that $W^\perp$ is not isotropic. Therefore $W^\perp$ must be symplectic and disjoint (except at the origin) from W. In other words, $\mathbb{R}^{2n} = W^\perp \oplus W$.

Choosing a symplectic basis (e_1, f_1) in $W^\perp$ we can assume, by a linear change of coordinates, that

$$W = \{(x, y) \,|\, x_1 = y_1 = 0\}.$$

Since Ω is bounded, there exists a real number N such that for $z = (x, y) \in \Omega + W$, $x_1^2 + y_1^2 < N^2$. That is, $\Omega + W \subset Z(N)$. Hence

$$c(\Omega + W) \leq c(Z(N)) = \pi N^2 < \infty.$$

The inequality $c > 0$ is always true, as we can always symplectically embed a small enough ball by Darboux's theorem.