The most important exercises are marked with an asterisk *.

6.1.

(a) Let (V, ω) be a symplectic vector space and $g\colon V \times V \to \mathbb{R}$ be an inner product. Show that there exists a symplectic basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ that is orthogonal with respect to g. Moreover, this basis can be chosen so that $g(e_j, e_j) = g(f_j, f_j)$.

Hint: Consider $\mathbb{R}^{2n}$ with the standard inner product and a linear symplectic form ω. Use an orthonormal basis $z_1, \ldots, z_n \in \mathbb{C}^n$ of eigenvectors of the skew-symmetric matrix A representing ω.

Solution. It is enough to consider $\mathbb{R}^{2n}$ with the standard inner product and a linear symplectic form ω. The 2-form ω can be represented by

$$\omega(v, w) = \langle v, Aw \rangle$$

for an invertible matrix A with $A^T = -A$. The skew-symmetry property implies that A is diagonalizable over $\mathbb{C}$. Its eigenvalues occur in pairs $\pm i\alpha_j$, $j = 1, \ldots, n$, with $\alpha_j > 0$. [1] Moreover, there exists an orthogonal (with respect to the hermitian product) basis of eigenvectors $z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n \in \mathbb{C}^{2n}$ with

$$Az_j = i\alpha_j z_j \text{ and } A\bar{z}_j = -i\alpha_j \bar{z}_j.$$

The orthogonality conditions read

$$\begin{aligned} \bar{z}_j^T z_k &= 0 \qquad \text{for } j \neq k, \\ z_j^T z_k &= 0 \qquad \text{for all } j, k. \end{aligned}$$

Writing $z_j = u_j + iv_j$ we get

$$\begin{aligned} &Au_j = -\alpha_j v_j, \qquad Av_j = \alpha_j u_j, \\ &u_j^T v_k = u_j^T u_k = v_j^T v_k = 0, \text{ for } j \neq k, \\ &\|u_j\|^2 = \|v_j\|^2, \qquad u_j^T v_j = 0. \end{aligned}$$

This implies

$$\omega(u_j, v_j) = u_j^T A v_j = \alpha_j \|u_j\|^2 > 0$$

and similarly

$$\omega(u_j, v_k) = \omega(u_j, u_k) = \omega(v_j, v_k) = 0$$

[1] These properties hold because iA is a hermitian matrix and hence iA is unitary diagonalizable. Its eigenvalues are real and come in pairs $\pm\alpha_j$ because iA is purely imaginary.

for $j \neq k$. Setting $e_j = \frac{v_j}{\sqrt{\alpha_j}\|v_j\|}$ and $f_j = \frac{u_j}{\sqrt{\alpha_j}\|u_j\|}$ we obtain an orthogonal symplectic basis. Moreover,

$$g(e_j, e_j) = g(f_j, f_j) = \frac{1}{\alpha_j}.$$

(b) Let g be an inner product on $\mathbb{R}^{2n}$ and consider the ellipsoid

$$E(g) = \left\{ w \in \mathbb{R}^{2n} \,|\, g(w,w) < 1 \right\}.$$

Show that there exists a symplectic linear matrix $A \in \mathrm{Sp}(2n)$ and an n-tuple $\boldsymbol{r} = (r_1, \ldots, r_n)$ with $0 < r_1 \leq \cdots \leq r_n$ and such that $AE = E(\boldsymbol{r})$, where

$$E(\boldsymbol{r}) = \left\{ (x,y) \in \mathbb{R}^{2n} \,\middle|\, \sum_{j=1}^{2n} \frac{x_j^2 + y_j^2}{r_j^2} < 1 \right\}.$$

Solution. We use a basis $e_1, \ldots, e_n, f_1, \ldots, f_n$ as in (a) for $\mathbb{R}^{2n}, \omega_{\mathrm{std}}$ and g. Let $r_1, \ldots, r_n > 0$ be given by

$$g(e_j, e_j) = g(f_j, f_j) = \frac{1}{r_j^2}.$$

By reordering the basis we can assume that $r_1 \leq r_2 \cdots \leq r_n$. Let $A \in \mathrm{Sp}(2n)$ be the symplectic matrix taking the standard basis of $\mathbb{R}^{2n}$ to $e_1, \ldots, e_n, f_1, \ldots, f_n$. Then

$$\begin{aligned} g(A(x,y), A(x,y)) &= g\left(\sum_{j=1}^n x_j e_j + \sum_{j=1}^n y_j f_j, \sum_{j=1}^n x_j e_j + \sum_{j=1}^n y_j f_j \right) \\ &= \sum_{j=1}^n \frac{x_j^2 + y_j^2}{r_j^2}. \end{aligned}$$

Thus

$$A^{-1}E(g) = E(A^*g) = E(\boldsymbol{r}).$$

(c) Show that the numbers $r_1, \ldots, r_n$ are uniquely determined by $E(g)$.

Hint: Suppose $E(\boldsymbol{r})$ and $E(\boldsymbol{s})$ are related by $A \in \mathrm{Sp}(2n)$. Show that $J_0 \,\mathrm{diag}(\frac{1}{r_1^2}, \ldots, \frac{1}{r_n^2})$ is similar to $J_0 \,\mathrm{diag}(\frac{1}{s_1^2}, \ldots, \frac{1}{s_n^2})$ and compare the eigenvalues.

Solution. The inner product that defines $E(\boldsymbol{r})$ is represented by the matrix

$$\Delta(\boldsymbol{r}) := \begin{pmatrix} \frac{1}{r_1^2} & 0 & \dots & 0 \\ 0 & \frac{1}{r_2^2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{r_n^2} \end{pmatrix}.$$

Suppose $A(E(\boldsymbol{s})) = E(\boldsymbol{r})$ for a linear symplectomorphism $A \in \mathrm{Sp}(2n)$. Then

$$A^T \Delta(\boldsymbol{r}) A = \Delta(\boldsymbol{s}).$$

We multiply this equation with J_0 from the left and use $J_0 A^T = A^{-1} J_0$ to get

$$A^{-1} J_0 \Delta(\boldsymbol{r}) A = J_0 \Delta(\boldsymbol{s}).$$

This shows that $J_0\Delta(\boldsymbol{s})$ is similar to $J_0\Delta(\boldsymbol{r})$. Hence these two matrices have the same eigenvalues. Since the eigenvalues are $\pm\frac{i}{s_1^2}, \dots, \pm\frac{i}{s_n^2}$, respectively $\pm\frac{i}{r_1^2}, \dots, \pm\frac{i}{r_n^2}$ it follows that $\boldsymbol{s} = \boldsymbol{r}$.

(d) Interpret the result for $n = 1$.

Solution. It says that any ellipse in $\mathbb{R}^2$ can be mapped into a circle by an area-preserving linear transformation. The radius of the circle is uniquely determined by the area constraint.

6.2. Let $E \subset \mathbb{R}^{2n}$ be an ellipsoid centered at 0. Show that there exists $A \in \mathrm{GL}(2n, \mathbb{R})$ such that $A^*\omega_{\mathrm{std}} = -\omega_{\mathrm{std}}$ and $A(E) = E$.

Solution. Consider the linear map $\Psi_0(x, y) = (-x, y)$. Clearly, $\Psi_0^*\omega_{\mathrm{std}} = -\omega_{\mathrm{std}}$. Given an ellipsoid E centered at 0, let $A \in \mathrm{Sp}(2n)$ such that $A(E) = E(\boldsymbol{r})$ as in 6.1.(b). Then $A^{-1}\Psi_0 A$ is anti-symplectic and

$$A^{-1}\Psi_0 A(E) = A^{-1}\Psi_0(E(\boldsymbol{r})) = A^{-1}(E(\boldsymbol{r})) = E.$$

***6.3.** Let $\psi_n \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be a sequence of continuous maps converging to a homeomorphism $\psi \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, uniformly on compact sets. Let $E \subset \mathbb{R}^{2n}$ be an ellipsoid centered at 0.

(a) Show that for any $\lambda < 1$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\psi_n(\lambda E) \subset \psi(E).$$

Solution. Consider $f_n := \psi^{-1} \circ \psi_n$. Then f_n converges to id uniformly on compact subsets. It follows that $f_n(\lambda E) \subset E$ for large enough n: Indeed, $\mathrm{id}(\overline{\lambda E}) \subset \overline{\lambda E}$ and hence $f_n(\overline{\lambda E})$ will eventually be contained in any neighbourhood of $\overline{\lambda E}$. In particular, $f_n(\overline{\lambda E}) \subset E$ for large enough λ. The inequality follows.

(b) Show that for any $\mu > 1$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\psi(E) \subset \psi_n(\mu E).$$

Hint: Consider maps $\phi_n \colon \mu\, \partial E \to S^{2n-1}$ obtained by normalizing $\psi^{-1} \circ \psi_n$ and study their degree.

Solution. Note that we can't use the same argument as in part (a) to get $E \subset f_n(E)$ for large enough n because f_n is not necessarily a homeomorphism. We have to work a bit harder here.

Since $f_n \to \mathrm{id}$ as in part (1), there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$f_n(\mu\, \partial E) \cap E = \emptyset$$

we can define the maps

$$\phi_n \colon \mu\, \partial E \to S^{2n-1}, \qquad x \mapsto \frac{f_n(x)}{\|f_n(x)\|}.$$

We claim that these maps have degree 1 for n large enough. To show this, we show that for sufficiently large n, ϕ_n are homotopic to

$$\phi \colon \mu\, \partial E \to S^{2n-1}, \qquad x \mapsto \frac{x}{\|x\|}.$$

From this it then follows that

$$\deg \phi_n = \deg \phi = 1$$

for n sufficiently large. Since f_n converges to the identity uniformly on compact sets, the restriction $f_n|_{\mu\partial E}$ converges uniformly to $\mathrm{id}\,|_{\mu\partial E}$. In particular, given a tubular neighbourhood U of $\mu\partial E$ with a deformation retraction $r \colon U \to \mu\partial E$, by choosing N large enough, we can make sure $f_n(\mu\partial E) \subset U$ for $n \geq N$. Moreover, for N large enough and $n \geq N$, the deformation retraction r gives us a homotopy $\{f_t\}_{t\in[0,1]}$ from $f_n|_{\mu\partial E}$ to $\mathrm{id}\,|_{\mu\partial E}$. To get a homotopy from ϕ_n to ϕ, we take

$$(s,x) \mapsto \frac{f_s(x)}{\|f_s(x)\|}.$$

This gives us the required homotopies and thus shows that ϕ_n has degree 1 for $n \geq N$.

We now show that $n \geq N$ it follows that $E \subset f_n(\mu E)$, which implies the statement to be proved. Suppose by contradiction that there exists $y_0 \in E$ such that for all $x \in \mu E$, $f_n(x) \neq y_0$. Then consider the maps

$$\begin{aligned} \phi_n' \colon \mu\,\partial E &\to S^{2n-1}, \\ x &\mapsto \frac{f_n(x) - y_0}{\|f_n(x) - y_0\|}. \end{aligned}$$

The map ϕ_n' extends to μE for $n \geq N$, hence it must have degree 0. On the other hand, ϕ_n' is homotopic to ϕ_n with a homotopy given by:

$$\phi_n^s \colon \mu\partial E \to S^{2n-1}, \qquad x \mapsto \frac{f_n(x) - s\cdot y_0}{\|f_n(x) - s\cdot y_0\|}$$

Thus the degrees of ϕ_n and ϕ_n' must coincide. This is a contradiction to ϕ_n having degree 1.

(c) Deduce that if ψ_n preserve the capacity of all ellipsoids, then also ψ preserves the capacity of all ellipsoids.

Solution. Let E be any ellipsoid centered at 0. Applying the capacity c to the inequalities from (a) and (b) we get

$$\lambda^2 c(E) = c(\lambda E) = c(\psi_n(\lambda E)) \leq c(E) \leq \psi_n(\mu E) = c(\mu E) = \mu^2 c(E),$$

for $\lambda < 1 < \mu$, where n is large enough. Taking limits as $\mu, \lambda \to 1$ yields $c(\psi(E)) = c(E)$. For an ellipsoid E not centered at 0 you consider a translation T such that $T(E)$ is centered at 0. Then apply what we proved to $T\psi_n T^{-1}$ and $T(E)$. It follows $c(T\psi T^{-1}(T(E))) = c(T(E))$. Using that c is a symplectic invariant we deduce $c(\psi(E)) = c(E)$.

***6.4.** Deduce from Exercise 6.3 that $\mathrm{Symp}(\mathbb{R}^{2n})$ is C^0-closed in $\mathrm{Diff}(\mathbb{R}^{2n})$.

Solution. Let $\psi_n \in \mathrm{Symp}(\mathbb{R}^{2n})$ be a sequence of symplectomorphisms converging in the C^0-topology to $\psi \in \mathrm{Diff}(\mathbb{R}^{2n})$. Then ψ_n preserves the capacity of ellipsoids for every n. By Exercise 6.3. ψ also preserves the capacity of ellipsoids. Hence by a theorem in lecture 11, ψ is either a symplectomorphism or an anti-symplectomorphism. Suppose $\psi^*\omega_{\text{std}} = -\omega_{\text{std}}$. Then $\phi_n := \psi_n \times \text{id} \in \text{Symp}(\mathbb{R}^{2n} \times \mathbb{R}^{2n}, \omega_{\text{std}} = \omega_{\text{std}} \times \omega_{\text{std}})$, while its C^0-limit $\phi := \psi \times \text{id}$ satisfies $\phi^*\omega_{\text{std}} = (-\omega_{\text{std}}) \times \omega_{\text{std}} \neq \pm\omega_{\text{std}}$. This is a contradiction to what we proved for C^0-limits of symplectomorphisms. Hence $\psi^*\omega_{\text{std}} = \omega_{\text{std}}$.