The most important exercises are marked with an asterisk *.

7.1. Consider two functions $F, G \in C^\infty(\mathbb{R}^{2n})$. The Poisson bracket of F and G is defined by

$$\{F, G\} := \sum_{j=1}^{n} \left(\frac{\partial F}{\partial x_j} \frac{\partial G}{\partial y_j} - \frac{\partial F}{\partial y_j} \frac{\partial G}{\partial x_j} \right) \in C^\infty(\mathbb{R}^{2n}).$$

*__(a)__ Show that $\{F, G\} = \omega_{\mathrm{std}}(X^G, X^F)$.

Solution. Using that ω_{std}, J_0 and the standard inner product form a compatible triple, $X^F = J_0 \nabla F$ and $X^G = J_0 \nabla G$, we compute

$$\begin{aligned} \omega_{\mathrm{std}}(X^G, X^F) &= \omega_{\mathrm{std}}(J_0 \nabla G, J_0 \nabla F) \\ &= \langle J_0 \nabla G, \nabla F \rangle \\ &= \left\langle \begin{pmatrix} \frac{\partial G}{\partial y} \\ -\frac{\partial G}{\partial x} \end{pmatrix}, \begin{pmatrix} \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \end{pmatrix} \right\rangle \\ &= \{F, G\}. \end{aligned}$$

*__(b)__ The function F is called an *integral* of the Hamiltonian differential equation

$$\dot{x}(t) = X^G(x(t))$$

if F is constant along its solutions $x(t)$. Show that F is an *integral* of the Hamiltonian differential equation associated to G if and only if $\{F, G\} = 0$.

Solution. We compute the derivative of $F(x(t))$ for a solution $x(t)$ of the Hamiltonian differential equation associated to G:

$$\frac{\mathrm{d}}{\mathrm{d}t} F(x(t)) = \mathrm{d}F_{x(t)} \left(X^G(x(t)) \right) = -\omega_{\mathrm{std}} \left(X^F(x(t)), X^G(x(t)) \right) = \{F, G\},$$

where we used (a) in the last equation. In particular, this derivative vanishes for all t, if and only if $\{F, G\} = 0$.

*__(c)__ Show that a diffeomorphism $\psi \colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is a symplectomorphism if and only if

$$\{F, G\} \circ \psi = \{F \circ \psi, G \circ \psi\}$$

for any functions $F, G \in C^\infty(M)$.

Solution. Old solutions containing a mistake: This also follows from (a):

$$\{F, G\} \circ \psi = \omega_{\mathrm{std}}(X^G \circ \psi, X^F \circ \psi)$$

and

$$\begin{aligned}\{F\circ\psi, G\circ\psi\} &= \omega_{\mathrm{std}}(X^{G\circ\psi}, X^{F\circ\psi})\\ &= \omega_{\mathrm{std}}((\mathrm{d}\psi)^{-1}(X^G\circ\psi), (\mathrm{d}\psi)^{-1}(X^F\circ\psi))\\ &= \left((\psi^{-1})^*\omega_{\mathrm{std}}\right)(X^G\circ\psi, X^F\circ\psi)\end{aligned}$$

These two terms coincide for all F, G if and only if $(\psi^{-1})^*\omega_{\mathrm{std}} = \omega_{\mathrm{std}}$. Namely, if and only if ψ is a symplectomorphism.

Explanation of the mistake: The two red colored terms in the equation are only equal if ψ is a symplectomorphism. Same for the two green colored terms. So this calculation shows actually only one direction of the implication: if ψ is a symplectomorphism then

$$\{F, G\}\circ\psi = \{F\circ\psi, G\circ\psi\}.$$

Correct solution: As in (a) we compute

$$\{F, G\}\circ\psi = \langle J_0\nabla G\circ\psi, \nabla F\circ\psi\rangle = -(\nabla G\circ\psi)^T J_0(\nabla F\circ\psi)$$

and

$$\begin{aligned}\{F\circ\psi, G\circ\psi\} &= -(\nabla(G\circ\psi))^T J_0\nabla(F\circ\psi)\\ &= -(\nabla G\circ\psi)^T\mathrm{d}\psi J_0(\mathrm{d}\psi)^T(\nabla F\circ\psi).\end{aligned}$$

Hence these two terms are equal for all F and G if and only if

$$\mathrm{d}\psi J_0(\mathrm{d}\psi)^T = J_0.$$

This in turn is equivalent to ψ being a symplectomorphism.

(d) Let X and Y be two symplectic vector fields. Show that $[X, Y]$ is a Hamiltonian vector field.

Hint: The bracket of two vector fields X and Y is defined by

$$[X, Y] = \nabla_X Y - \nabla_Y X.$$

Use the formula $\iota_{[X,Y]}\omega = \mathcal{L}_X(\iota_Y\omega) - \iota_Y(\mathcal{L}_X\omega)$ to show that $\iota_{[X,Y]}\omega_{\mathrm{std}}$ is exact.

Solution. We abbreviate $\omega = \omega_{\mathrm{std}}$. We proceed as in the hint, using Cartan's formula twice:

$$\begin{aligned}\iota_{[X,Y]}\omega &= \mathcal{L}_X(\iota_Y\omega) - \iota_Y(\mathcal{L}_X\omega)\\ &= \mathrm{d}\iota_X(\iota_Y\omega) + \iota_X\mathrm{d}(\iota_Y\omega) - \iota_Y(\mathrm{d}\iota_X\omega) - \iota_Y(\iota_X\mathrm{d}\omega)\\ &= \mathrm{d}\omega(Y, X),\end{aligned}$$

where in the last equation we used that $\mathrm{d}(\iota_Y\omega) = 0$ and $\mathrm{d}\iota_X\omega = 0$ because Y, X are symplectic. Hence $\iota_{[X,Y]}\omega$ is exact and in fact $\omega_{\text{std}}(X, Y)$ is a Hamiltonian function generating the vector field $[X, Y]$.

(e) Show that

$$[X^F, X^G] = X^{\{G,F\}}.$$

Solution. This follows from (a) and what we've shown in (d). Indeed, in (d), we showed that $[X^F, X^G]$ is a Hamiltonian vector field corresponding to the Hamiltonian $\omega_{\text{std}}(X^F, X^G)$. Thus:

$$[X^F, X^G] = X^{\omega_{\text{std}}(X^F,X^G)} = X^{\{G,F\}}.$$

***7.2.** Let c be a symplectic capacity. Define

$$\check{c}(M, \omega) := \sup\{c(U, \omega) \,|\, U \subset M \text{ open}, \overline{U} \subset M\backslash\partial M\}.$$

We always have $\check{c} \leq c$. The capacity c is called *inner regular* if $c = \check{c}$. Show:

(a) Corrected version!!! The measurement $\check{c}$ does satisfy the conformality and non-triviality axiom, but it is not necessarily a symplectic capacity.

Remark: Changing the definition of $\check{c}$ by taking supremum only over $c(U, \omega)$ where $\overline{U}$ is in addition compact, will result in an actual symplectic capacity.

Solution.

The measurement $\check{c}$ does satisfy the conformality and non-triviality axiom, but not necessarily the monotonicity axiom:

- **Monotonicity.** We show why the obvious proof fails:

  Let $\varphi\colon (M, \omega) \to (N, \tau)$ be a symplectic embedding. Then for every $U \subset M$ open and such that $\overline{U} \subset M \setminus \partial M$, the map $\varphi|_U$ is also a symplectic embedding. Therefore, since c is a capacity, it holds $c(U, \omega) \leq c(\varphi(U), \tau)$. Taking supremum over U, we get

  $$\sup_{\substack{U \text{open} \\ \overline{U}\subset M\backslash\partial M}} c(U, \omega) \leq \sup_{\substack{U \text{open} \\ \overline{U}\subset M\backslash\partial M}} c(\varphi(U), \tau).$$

  The problem is that $V := \varphi(U)$ could intersect the boundary of N, so that we can't conclude that

  $$\sup_{\substack{U \text{open} \\ \overline{U}\subset M\backslash\partial M}} c(\varphi(U), \tau) \leq \sup_{\substack{V \text{open} \\ \overline{V}\subset N\backslash\partial N}} c(V, \tau) = \check{c}(N, \omega).$$

To see that V could intersect the boundary of N, consider the symplectic embedding φ of the open unit ball M into the closed unit ball N. Then $U = M$ is an open set as it occurs in the definition of $\check{c}(M)$. However, the closure of $\varphi(U)$ is N, hence intersects the boundary of N.

- **Conformality.** We need to show that $\check{c}(M, \alpha\omega) = |\alpha| \cdot \check{c}(M, \omega)$ for all $\alpha \in \mathbb{R} \setminus \{0\}$. Since c is a symplectic capacity, it follows that $c(U, \alpha\omega) = |\alpha| \cdot c(U, \alpha)$. Taking supremums over all open U with $\overline{U} \subset M \setminus \partial M$ shows conformality.

- **Non-triviality.** We show that $\check{c}(B(1), \omega_0) = \pi$. Since $(B(1), \omega_0)$ embeds in $(B(1), \omega_0)$, it must hold that $\check{c}(B(1), \omega_0) \leq \pi$. Suppose for contradiction that $\check{c}(B(1), \omega_0) = \pi\varepsilon^2$ for some $\varepsilon \in (0, 1)$. Note that for $\delta = (1 + \varepsilon)/2$ we can symplectically embed $(B(\delta), \omega_0)$ into $(B(1), \omega_0)$. Thus

$$\check{c}(B(1), \omega_0) \geq c(B(\delta), \omega_0) = \left(\frac{1+\varepsilon}{2}\right)^2 \pi > \varepsilon^2 \pi.$$

  This is a contradiction to our assumption that $\check{c}(B(1), \omega_0) = \pi\varepsilon^2$ and thus the inner capacity of the ball must be $\check{c}(B(1), \omega_0) = \pi$.

  The argument for $Z(1)$ is analogous.

(b) If d is any inner regular symplectic capacity with $d \leq c$, then $d \leq \check{c}$.

Solution. Since d is inner regular and $d \leq c$, we have

$$\begin{aligned} d(M) = \check{d}(M) &= \sup\{d(U, \omega) \,|\, U \subset M \text{ open }, \overline{U} \subset M \backslash \partial M\} \\ &\leq \sup\{c(U, \omega) \,|\, U \subset M \text{ open }, \overline{U} \subset M \backslash \partial M\} \\ &= \check{c}. \end{aligned}$$

(c) The Gromov-width $D(M, \omega)$ is inner regular.

Solution. Let $0 < a < D(M, \omega)$. Then for $\epsilon > 0$ satisfying $a + \epsilon < D(M, \omega)$, there exists a symplectic embedding

$$\varphi\colon B(a + \epsilon) \hookrightarrow (M, \omega).$$

But then $U \coloneqq \varphi(B(a)) \subset M$ satisfies $\overline{U} \subset M \backslash \partial M$. In particular, $D(U, \omega) \geq a$ and thus $\check{D}(M, \omega) \geq a$. Since this is true for any $0 < a < D(M, \omega)$ we conclude $\check{D}(M, \omega) = D(M, \omega)$.

(d) The Hofer-Zehnder capacity c_0 is inner regular.

Solution. Let (M,ω) be a symplectic manifold, possibly with boundary. Suppose $c_0(M,\omega) < \infty$. For $\epsilon > 0$ there exists $H \in \mathcal{H}_a(M,\omega)$ such that

$$m(H) > c_0(M,\omega) - \epsilon.$$

Let $K \subset M$ be the support of X^H. By property (1) for elements in the set $\mathcal{H}(M,\omega)$, we have $K \subset M \backslash \partial M$. Pick an open set U with $K \subset U \subset \overline{U} \subset M \backslash \partial M$. Clearly, the restriction $H|_U$ is contained in $\mathcal{H}_a(U,\omega)$ and $m\left(H|_U\right) = m(H)$. Thus $c_0(U,\omega) \geq m(H) > c_0(M,\omega) - \epsilon$. It follows that $\check{c}_0(M,\omega) \geq c_0(M,\omega)$. The other inequality is clear.

***7.3.** What is the biggest symplectic capacity?

Solution. Similar to the Gromov-width, that measures the size of the biggest ball that embeds into a symplectic manifold M, we can measure the smallest size of a cylinder into which we can squeeze M. More precisely, we define

$$\widetilde{D}(M,\omega) := \inf\{\pi r^2 \,\big|\, \exists\, M \xhookrightarrow{\;s\;} Z(r)\}.$$

We show that $\widetilde{D}$ satisfies the monotonicity axiom: Let $\psi\colon (M,\omega) \hookrightarrow (N,\sigma)$ be a symplectic embedding. Note that any symplectic embedding $\varphi\colon N \hookrightarrow Z(r)$ gives a symplectic embedding $\varphi \circ \psi\colon M \hookrightarrow Z(r)$. Therefore,

$$\begin{aligned}
\widetilde{D}(N,\sigma) &= \inf\{\pi r^2 \,\big|\, \exists\, N \xhookrightarrow{\;s\;} Z(r)\} \\
&\geq \inf\{\pi r^2 \,\big|\, \exists\, M \xhookrightarrow{\;s\;} Z(r)\} \\
&= \widetilde{D}(M,\omega).
\end{aligned}$$

For the conformality axiom, we need to show that

$$\widetilde{D}(M,\alpha\omega) = |\alpha|\widetilde{D}(M,\omega)$$

It is enough to construct a bijection between

$$\left\{\varphi\colon (M,\alpha\omega) \overset{s}{\hookrightarrow} Z(r)\right\}$$

and

$$\left\{\hat{\varphi}\colon (M,\omega) \overset{s}{\hookrightarrow} Z\left(\frac{r}{\sqrt{|\alpha|}}\right)\right\}$$

for any real number $\alpha \neq 0$. Consider

$$f \colon Z(r) \to Z\left(\frac{r}{\sqrt{|\alpha|}}\right),\, x \mapsto \frac{x}{\sqrt{|\alpha|}}.$$

If $\alpha > 0$, we get a bijection by setting $\hat{\varphi} = f \circ \varphi$. Indeed, $\hat{\varphi}$ is a symplectic embedding:

$$\hat{\varphi}^*\omega_{\mathrm{std}} = \varphi^* f^* \omega_{\mathrm{std}} = \varphi^*(\alpha^{-1}\omega_{\mathrm{std}}) = \alpha^{-1}\alpha\omega = \omega.$$

If $\alpha < 0$ a bijection is given by $\hat{\varphi} = \psi_0 \circ f \circ \varphi$, where $\psi_0(u,v) = (-u,v)$ is an anti-symplectomorphism on $Z(r)$. Then $\hat{\varphi}$ is again a symplectic embedding:

$$\hat{\varphi}^*\omega_{\mathrm{std}} = \varphi^* f^* \psi_0^* \omega_{\mathrm{std}} = -\varphi^* f^* \omega_{\mathrm{std}} = -\varphi^*(|\alpha|^{-1}\omega_{\mathrm{std}}) = \alpha^{-1}\alpha\omega = \omega.$$

Finally the non-triviality axiom follows from non-squeezing: $\widetilde{D}(B(1), \omega_{\mathrm{std}}) = \pi$ because there exists a symplectic embedding $B(1) \overset{s}{\hookrightarrow} Z(r)$ if and only if $1 \leq r$. $\widetilde{D}(Z(1), \omega_{\mathrm{std}}) = \pi$ follows from the definition of $\widetilde{D}$.

This shows that $\widetilde{D}$ is a symplectic capacity. Moreover, any symplectic capacity c is bounded from above by $\widetilde{D}$. Indeed, we have

$$\pi r^2 = c(Z(r), \omega_{\mathrm{std}}) \geq c(M, \omega),$$

whenever $(M,\omega) \overset{s}{\hookrightarrow} (Z(r), \omega_{\mathrm{std}})$. Hence $\widetilde{D}(M,\omega) \geq c(M,\omega)$.

7.4. Let $H \in C^\infty(\mathbb{R} \times \mathbb{R}^{2n})$ be a Hamiltonian function that is 1-periodic: $H_t = H_{t+1}$ for any t. On a loop $z \in C^\infty(S^1, \mathbb{R}^{2n})$, the action functional takes the value

$$\mathcal{A}_H(z) = \int_0^1 \frac{1}{2}\langle -J_0 \dot{z}(t), z(t)\rangle \,\mathrm{d}t - \int_0^1 H_t(z(t))\,\mathrm{d}t.$$

Show that this coincides with the physicist's action functional, namely for a loop $z(t) = (x(t), y(t))$ we have

$$\mathcal{A}_H(z) = \int_0^1 \langle y(t), \dot{x}(t)\rangle \,\mathrm{d}t - \int_0^1 H_t(z(t))\,\mathrm{d}t.$$

In other words, $\mathcal{A}_H(z)$ is the integral of the *action* 1-*form*

$$\lambda_H := \sum_{j=1}^n y_j \mathrm{d}x_j - H\mathrm{d}t$$

along the loop z.

Hint: Integration by parts.

Solution. It's only the first term we need to study. We have

$$\begin{aligned}\frac{1}{2}\langle -J_0\dot{z}(t), z(t)\rangle &= \frac{1}{2}\left\langle \begin{pmatrix}-\dot{y}(t)\\ \dot{x}(t)\end{pmatrix}, \begin{pmatrix}x(t)\\ y(t)\end{pmatrix}\right\rangle \\ &= \frac{1}{2}\left(\langle y(t), \dot{x}(t)\rangle - \langle \dot{y}(t), x(t)\rangle\right)\end{aligned}$$

Integration by parts implies

$$\begin{aligned}\int_0^1 \frac{1}{2}\langle -J_0\dot{z}(t), z(t)\rangle \,\mathrm{d}t &= \frac{1}{2}\int_0^1 \left(\langle y(t), \dot{x}(t)\rangle - \langle \dot{y}(t), x(t)\rangle\right)\mathrm{d}t \\ &= \frac{1}{2}\left(\int_0^1 \langle y(t), \dot{x}(t)\rangle \mathrm{d}t - \langle y(t), x(t)\rangle\Big|_0^1 + \int_0^1 \langle y(t), \dot{x}(t)\rangle \mathrm{d}t\right) \\ &= \int_0^1 \langle y(t), \dot{x}(t)\rangle \mathrm{d}t.\end{aligned}$$

Here the last equation follows because $y(0) = y(1)$ and $x(0) = x(1)$.

The last assertion follows directly from the definition of the integral of a 1-form:

$$\begin{aligned}\int_{S^1} z^*\lambda_H &= \int_0^1 (\lambda_H)_{z(t)}(\dot{z}(t))\mathrm{d}t \\ &= \int_0^1 \sum_{j=1}^n y_j(t)\dot{x}_j(t)\mathrm{d}t - \int_0^1 H_t(z(t))\mathrm{d}t \\ &= \int_0^1 \langle y(t), \dot{x}(t)\rangle \mathrm{d}t - \int_0^1 H_t(z(t))\mathrm{d}t \\ &= \mathcal{A}_H(z).\end{aligned}$$