The most important exercises are marked with an asterisk *.

***8.1.** Let $N \in \mathbb{N}$, $\pi < a < 2\pi$ and consider the quadratic Hamiltonian $Q\colon \mathbb{R}^{2n} \to \mathbb{R}$ defined by

$$Q(x,y) = a(x_1^2 + y_1^2) + \frac{a}{N^2}\sum_{j=2}^{n}(x_j^2 + y_j^2).$$

Show that there are no 1-periodic solutions to $\dot{z} = X^Q(z)$ except the constant solution $z = 0$.

Solution.

We write $z = (x, y)$ and consider a solution with $z(0) \neq 0$. Then since, as we saw in lectures, $\dot{z}(t) = J_0 \nabla Q(z)$, it follows that

$$\begin{pmatrix} -\dot{y}(t) \\ \dot{x}(t) \end{pmatrix} = -J_0\dot{z} = \nabla Q(z) = 2a\,\mathrm{diag}\left(1, \frac{1}{N^2}, \dots, \frac{1}{N^2}, 1, \frac{1}{N^2}, \dots, \frac{1}{N^2}\right) z.$$

Therefore, the ODE decouples into n ODE's:

$$(-\dot{y}_1, \dot{x}_1) = 2a(x_1, y_1)$$

and for $2 \leq k \leq n$

$$(-\dot{y}_k, \dot{x}_k) = \frac{2a}{N^2}(x_k, y_k).$$

Therefore, writing $z_1 = (x_1, y_1)$ we get

$$z_1(t) = \cos(2at)z_1(0) + \sin(2at)J_0 z_1(0)$$

and for $2 \leq k \leq n$, $z_k = (x_k, y_k)$ we get

$$z_k(t) = \cos\left(\frac{2a}{N^2}t\right) z_k(0) + \sin\left(\frac{2a}{N^2}t\right) J_0 z_k(0).$$

Thus $z(t)$ have smallest period $T = \frac{\pi N^2}{a}$. If $N \geq 2$ we see $T > \frac{\pi N^2}{2\pi} \geq 2$. If $N = 1$ we have $T = \frac{\pi}{a} < 1$. In both cases, $T \neq 1$ which proves the result.

***8.2.** Let E be a Hilbert space and $f \in C^1(E, \mathbb{R})$. A subset $R \subset E$ is called a mountain range for f, if

- $E \backslash R$ is disconnected,

- $\alpha \coloneqq \inf_R f > -\infty$,
- and on every component of $E \backslash R$, the function f attains a value strictly less than α.

Prove the Mountain Pass Lemma: Assume that f satisfies the Palais-Smale condition and assume that the gradient equation

$$\dot{x} = -\nabla f(x)$$

generates a global flow on E. Then for any mountain range $R \subset E$, the function f has a critical point $x \in E$ satisfying $f(x) \geq \alpha$.

Solution. Choose two different components E^0 and E^1 of $E \backslash R$. Let

$$\Gamma \coloneqq \{\gamma \in C([0,1], E) \,\big|\, \gamma(0) \in E^0, \gamma(1) \in E^1, f(\gamma(0)), f(\gamma(1)) < \alpha\}$$

and

$$\mathcal{F} \coloneqq \left\{\mathrm{im}(\gamma) \,\middle|\, \gamma \in \Gamma\right\}.$$

Consider $\gamma \in \Gamma$. Then $\gamma(0)$ and $\gamma(1)$ lie in two different components of $E \backslash R$ and hence there exists $t_0 \in (0,1)$ such that $\gamma(t_0) \in R$. In particular,

$$\sup_{t \in [0,1]} f(\gamma(t)) \geq f(\gamma(t_0)) \geq \alpha.$$

Thus

$$c(f, \mathcal{F}) = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} f(\gamma(t)) \geq \alpha.$$

Clearly, $c(f, \mathcal{F}) < \infty$ because f attains its maximum on the compact set $\mathrm{im}(\gamma)$. Moreover, $\mathcal{F}$ is positively invariant under the flow φ_t of $-\nabla f$: for $\gamma \in \Gamma$ and $t \geq 0$ we have $\varphi_t \circ \gamma \in \Gamma$ because f decreases along flow lines. Thus f and $\mathcal{F}$ satisfy all assumptions of the minimax lemma. Therefore, $c(f, \mathcal{F}) \geq \alpha$ is a critical value of f.

8.3. Show that H^1 is a Hilbert space. You should use the fact that $L^2(S^1)$ is a Hilbert space and the definition of H^1 using Fourier series.

Hint: Given a Cauchy sequence $x^n \in H^1$, consider the sequences x^n and its weak derivative $y^n \coloneqq \dot{(x^n)}$ as Cauchy sequences in $L^2(S^1)$.

Solution. We need to show that H^1 is complete. Let $x^n \in H^1$ be a Cauchy sequence. Write

$$x^n = \sum_{k \in \mathbb{Z}} e^{2\pi k J_0 t} x_k^n$$

with $x_k^n \in \mathbb{R}^{2n}$. We have

$$\|x^n - x^m\|_0^2 \leq \|x^n - x^m\|_1^2$$

and thus x^n is a Cauchy sequence wrt to $\|\cdot\|_0$. Consider the sequence

$$y^n := 2\pi \sum_{k\in\mathbb{Z}} e^{2\pi k J_0 t}(kJ_0 x_k^n).$$

We have

$$\begin{aligned}
\|y^n - y^m\|_0^2 &= \|y_0^n - y_0^m\|^2 + 2\pi \sum_{k\in\mathbb{Z}} \|y_k^n - y_k^m\|^2 \\
&= 0 + 2\pi \sum_{k\in\mathbb{Z}} 4\pi^2 k^2 \|x_k^n - x_k^m\|^2 \\
&\leq 4\pi^2 \|x^n - x^m\|_1^2 .
\end{aligned}$$

Thus also y^n is a Cauchy sequence for $\|\cdot\|_0$. Since $\|\cdot\|_0$ is equivalent to $\|\cdot\|_{L^2}$ on $H^0 = L^2(S^1)$ and $L^2(S^1)$ is complete, x^n converges to some $x \in L^2(S^1)$ and y_n converges to some $y \in L^2(S^1)$ wrt $\|\cdot\|_{L^2}$.

Since $\|y^n - y\|_0 \to 0$, we have $\|2\pi k J_0 x_k^n - y_k\| \to 0$ and thus

$$y_k = \lim_{n\to\infty} 2\pi k J_0 x_k^n = 2\pi k J_0 x_k.$$

We therefore have

$$\begin{aligned}
\|x^n - x\|_1^2 &= \|x_0^n - x_0\|^2 + 2\pi \sum_{k\in\mathbb{Z}} k^2 \|x_k^n - x_k\|^2 \\
&= \|x_0^n - x_0\|^2 + 2\pi \sum_{k\in\mathbb{Z}} k^2 \frac{1}{4\pi^2 k^2} \|y_k^n - y_k\|^2 \\
&\leq C \|y^n - y\|_0^2 \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Therefore, x^n converges to x in $\|\cdot\|_1$. Moreover,

$$\|x\|_1 \leq \|x - x_n\|_1 + \|x_n\|_1 < \infty,$$

so that $x \in H^1$.

8.4. If you don't know the Fourier series representation for elements in $L^2(S^1)$ and you are interested in it, read it up for example in sections 1.1 and 1.2 in `https://math.iisc.ac.in/~veluma/fourier.pdf`