The most important exercises are marked with an asterisk *.

***9.1.** Let $H\colon [0,1]\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ be a smooth compactly supported Hamiltonian function on $\mathbb{R}^{2n}$. Let $x(t)$ be a 1-periodic solution to $\dot{x}(t) = X^H(x(t))$. The goal of this exercise is to show that the action

$$\mathcal{A}_H(x) = \int_0^1 \frac{1}{2}\langle -J_0\dot{x}(t), x(t)\rangle - \int_0^1 H_t(x(t))\,\mathrm{d}t.$$

only depends on $x(0)$ and ψ_1^H and not on H.

(a) Let H and K be Hamiltonians as above and assume $\psi_1^H = \psi_1^K$. Consider the piecewise smooth path $t\mapsto\varphi_t$ defined by

$$\varphi_t = \begin{cases} \psi_t^H & t\in[0,1],\\ \psi_{2-t}^K & t\in[1,2]. \end{cases}$$

Let $x_0\in\mathbb{R}^{2n}$ and define $\Delta(x_0)$ to be the action of the loop $x(t)=\varphi_t(x_0)$:

$$\Delta(x_0) = \int_0^2 \frac{1}{2}\langle -J_0\dot{x}(t), x(t)\rangle\,\mathrm{d}t - \int_0^1 H_t(x(t))\,\mathrm{d}t + \int_1^2 K_{2-t}(x(t))\,\mathrm{d}t.$$

Show that

$$\Delta(x_0) = \mathcal{A}_H(\psi_t^H(x_0)) - \mathcal{A}_K(\psi_t^K(x_0))$$

if $\psi_1^H(x_0) = x_0$.

Solution. This is a direct computation:

$$\begin{aligned}
\Delta(x_0) &= \int_0^2 \frac{1}{2}\langle -J_0\dot{x}(t), x(t)\rangle\,\mathrm{d}t - \int_0^1 H_t(x(t))\,\mathrm{d}t + \int_1^2 K_{2-t}(x(t))\,\mathrm{d}t\\
&= \int_0^1 \frac{1}{2}\left\langle -J_0\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^H(x_0)\right), \psi_t^H(x_0)\right\rangle\mathrm{d}t + \int_1^2 \frac{1}{2}\left\langle -J_0\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_{2-t}^K(x_0)\right), \psi_{2-t}^K(x_0)\right\rangle\mathrm{d}t\\
&\quad - \int_0^1 H_t(\psi_t^H(x_0))\,\mathrm{d}t + \int_1^2 K_{2-t}(\psi_{2-t}^K(x_0))\,\mathrm{d}t\\
&= \int_0^1 \frac{1}{2}\left\langle -J_0\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^H(x_0)\right), \psi_t^H(x_0)\right\rangle\mathrm{d}t - \int_0^1 H_t(\psi_t^H(x_0))\,\mathrm{d}t\\
&\quad - \int_0^1 \frac{1}{2}\left\langle -J_0\frac{\mathrm{d}}{\mathrm{d}s}\left(\psi_s^K(x_0)\right), \psi_s^K(x_0)\right\rangle\mathrm{d}s + \int_0^1 K_s(\psi_s^K(x_0))\,\mathrm{d}t\\
&= \mathcal{A}_H(\psi_t^H(x_0)) - \mathcal{A}_K(\psi_t^K(x_0)),
\end{aligned}$$

where we used the substitution $s = 2-t$ in the third equality.

(b) Show that Δ is differentiable and $\mathrm{d}\Delta_x = 0$.

Solution. For $x_0 \in \mathbb{R}^{2n}$ we write as before $x(t) = \varphi_t(x_0)$. Let $h \in \mathbb{R}^{2n}$. Using the Leibniz rule we compute the derivative of the first part:

$$\begin{aligned}
&\int_0^2 \frac{1}{2}\left\langle -J_0 \mathrm{d}\left(\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x_0)\right)_{x_0}(h), \varphi_t(x)\right\rangle \mathrm{d}t + \int_0^2 \frac{1}{2}\left\langle -J_0 \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x_0), (\mathrm{d}\varphi_t)_{x_0}(h)\right\rangle \mathrm{d}t \\
&= \int_0^2 \frac{1}{2}\left\langle \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{d}\varphi_t)_{x_0}(h), J_0\varphi_t(x)\right\rangle \mathrm{d}t + \int_0^2 \frac{1}{2}\left\langle -J_0 \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x_0), (\mathrm{d}\varphi_t)_{x_0}(h)\right\rangle \mathrm{d}t \\
&= -\int_0^2 \frac{1}{2}\left\langle (\mathrm{d}\varphi_t)_{x_0}(h), \frac{\mathrm{d}}{\mathrm{d}t}J_0\varphi_t(x)\right\rangle \mathrm{d}t + \int_0^2 \frac{1}{2}\left\langle -J_0 \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x_0), (\mathrm{d}\varphi_t)_{x_0}(h)\right\rangle \mathrm{d}t \\
&= \int_0^2 \left\langle -J_0 \frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(x_0), (\mathrm{d}\varphi_t)_{x_0}(h)\right\rangle \mathrm{d}t \\
&= \int_0^2 \langle -J_0\dot{x}(t), (\mathrm{d}\varphi_t)_{x_0}(h)\rangle \,\mathrm{d}t.
\end{aligned}$$

Here, in the first equation we exchanged the order of differentiation in the first integral and moved J_0 to the other side ($J_0^T = -J_0$). Then we use integration by parts and the fact that

$$\int_0^2 \frac{\mathrm{d}}{\mathrm{d}t}\langle (\mathrm{d}\varphi_t)_{x_0}(h), J_0\varphi_t(x)\rangle \,\mathrm{d}t = \langle (\mathrm{d}\varphi_t)_{x_0}(h), J_0\varphi_t(x)\rangle|_0^2 = 0.$$

We now compute the derivative of Δ:

$$\begin{aligned}
\mathrm{d}\Delta_{x_0}(h) &= \int_0^2 \langle -J_0\dot{x}(t), (\mathrm{d}\varphi_t)_{x_0}(h)\rangle \,\mathrm{d}t - \int_0^1 \langle \nabla H_t(x(t)), (\mathrm{d}\varphi_t)_{x_0}(h)\rangle \,\mathrm{d}t \\
&\quad + \int_1^2 \langle \nabla K_{2-t}(x(t)), (\mathrm{d}\varphi_t)_{x_0}(h)\rangle \,\mathrm{d}t \\
&= \int_0^1 \langle -J_0\dot{x}(t) - \nabla H_t(x(t)), (\mathrm{d}\varphi_t)_{x_0}(h)\rangle \,\mathrm{d}t \\
&\quad + \int_1^2 \langle -J_0\dot{x}(t) + \nabla K_{2-t}(x(t)), (\mathrm{d}\varphi_t)_{x_0}(h)\rangle \,\mathrm{d}t
\end{aligned}$$

The first term vanishes because for $t \in [0,1]$, we have $x(t) = \psi_t^H(x_0)$, hence

$$-J_0\dot{x}(t) = \nabla H_t(x(t)).$$

For the second term, note that $x(t) = \psi_{2-t}^K(x_0)$ for $t \in [1,2]$. Therefore, by the chain rule,

$$-J_0\dot{x}(t) = -J_0\frac{\mathrm{d}}{\mathrm{d}t}\psi_{2-t}^K(x_0) = -\nabla K_{2-t}(\psi_{2-t}^K(x_0)) = -\nabla K_{2-t}(x(t)).$$

In particular, the second integral vanishes as well. All together we conclude $\mathrm{d}\Delta_{x_0}(h) = 0$.

(c) Conclude that

$$\mathcal{A}_H(\psi_t^H(x_0)) = \mathcal{A}_K(\psi_t^K(x_0))$$

if $\psi_1^H(x_0) = x_0$.

Solution. Since H and K are compactly supported, there exists a compact subset $V \subset \mathbb{R}^{2n}$ such that $H_t \equiv K_t \equiv 0$ outside of V. Then for $x_0 \notin V$, we have $\psi_t^H(x_0) = \psi_t^K(x_0) = x_0$ and hence $\varphi_t(x_0) = x_0$ for all t. It follows that $\Delta(x_0) = 0$. By part (b) we know that Δ is constant. We conclude that $\Delta(x_0) = 0$ for each $x_0 \in \mathbb{R}^{2n}$. The result now follows from Δ being the difference of the two action functionals as stated in part (a).

We can therefore define

$$\mathcal{A}(x, \varphi) := \mathcal{A}_H(\psi_t^H(x))$$

for $\varphi = \psi_1^H \in \mathrm{Ham}(\mathbb{R}^{2n})$ and any fixed point x of φ.

***9.2.** Let $\vartheta \in \mathrm{Symp}(\mathbb{R}^{2n})$, $\varphi \in \mathrm{Ham}_c(\mathbb{R}^{2n})$ a Hamiltonian diffeomorphism generated by a compactly supported Hamiltonian and $x \in \mathbb{R}^{2n}$ a fixed point of φ. Show that $\mathcal{A}(\vartheta(x), \vartheta\varphi\vartheta^{-1}) = \mathcal{A}(x, \varphi)$.

Hint: The following exercises are useful: Exercises 2.2., 3.2(b) and 7.4.

Solution. First note that $x \in \mathbb{R}^{2n}$ is a fixed point of φ if and only if $\vartheta(x)$ is a fixed point of $\vartheta\varphi\vartheta^{-1}$.

Let H be a compactly supported Hamiltonian such that $\psi_1^H = \varphi$. We set $x(t) = \psi_t^H(x)$. By Exercise 2.2.(c), $\vartheta\varphi\vartheta^{-1}$ is a Hamiltonian diffeomorphism generated by $K_t = H_t \circ \vartheta^{-1}$ and we have $y(t) := \psi_t^K(\vartheta(x)) = \vartheta\psi_t^H(x) = \vartheta(x(t))$.

Denote by $\lambda := \sum_{j=1}^n y_j \mathrm{d}x_j$ the canoncial 1-form. Then the 1-form $\vartheta^*\lambda - \lambda$ is closed because

$$\mathrm{d}(\vartheta^*\lambda - \lambda) = \vartheta^*\mathrm{d}\lambda - \mathrm{d}\lambda = \vartheta^*\omega_{\mathrm{std}} - \omega_{\mathrm{std}} = 0.$$

Since any closed 1-form on $\mathbb{R}^{2n}$ is exact (Exercise 3.2.(b)) we deduce that $\vartheta^*\lambda - \lambda = \mathrm{d}F$ for a function $F\colon \mathbb{R}^{2n} \to \mathbb{R}$.

We now use the alternative formula for the action functional from Exercise 7.4 to compute

$$\begin{aligned}
\mathcal{A}(\vartheta(x), \vartheta\varphi\vartheta^{-1}) &= \int_{S^1} y^*\lambda - \int_0^1 K_t(y(t))\,\mathrm{d}t \\
&= \int_{S^1} x^*\vartheta^*\lambda - \int_0^1 (H_t\vartheta^{-1})(\vartheta(x(t)))\,\mathrm{d}t \\
&= \int_{S^1} x^*(\lambda + \mathrm{d}F) - \int_0^1 H_t(x(t))\,\mathrm{d}t \\
&= \int_{S^1} x^*\lambda + \int_{S^1} \mathrm{d}(x^*F) - \int_0^1 H_t(x(t))\,\mathrm{d}t \\
&= \int_{S^1} x^*\lambda - \int_0^1 H_t(x(t))\,\mathrm{d}t \\
&= \mathcal{A}(x, \varphi),
\end{aligned}$$

where in the second last identity we used Stokes.