The most important exercises are marked with an asterisk *.

***10.1.** Let (M, ω) and (M', ω') be a symplectic manifolds of the same dimension. Let $\psi\colon M \to M'$ be a smooth map and consider its graph

$$\Gamma_\psi \coloneqq \{(x, \psi(x)) \,|\, x \in M\} \subset (M \times M', \omega \oplus (-\omega')).$$

Show that ψ is symplectic (i.e. $\psi^*\omega' = \omega$) if and only if Γ_ψ is a Lagrangian submanifold.

Solution. Note first that $\dim \Gamma_\psi = \dim M = \frac{1}{2}\dim(M \times M')$. Thus Γ_ψ is a Lagrangian if and only if the restriction of $\omega \oplus (-\omega')$ to TL vanishes. Consider the graph map

$$gr_\psi\colon M \to M \times M',\ x \mapsto (x, \psi(x)).$$

This is a diffeomorphism onto Γ_ψ and hence the condition of being a Lagrangian is equivalent to $gr_\psi^*(\omega \oplus (-\omega')) = 0$. We have

$$gr_\psi^*(\omega \oplus (-\omega')) = \omega - \psi^*\omega'.$$

This vanishes if and only if $\omega = \psi^*\omega'$.

***10.2.** Let Q be a smooth manifold and $S \subset Q$ a submanifold. Prove that the annihilator

$$TS^0 \coloneqq \{(x, \xi) \in T^*Q \,|\, x \in S, \xi|_{T_xS} \equiv 0\}$$

is a Lagrangian submanifold of T^*Q with respect to the canonical symplectic form.

Solution. First we show that TS^0 is submanifold of T^*Q of dimension $n \coloneqq \dim Q$. Denote by $m \coloneqq \dim S$. Let $U \subseteq Q$ be an open subset and $\varphi\colon U \to \mathbb{R}^n$ a chart such that $\varphi(U \cap S) = \mathbb{R}^m \times \{0\}$. This induces a chart

$$\Phi\colon T^*U \to T^*\mathbb{R}^n,\ \Phi(x, \xi) = (\varphi(x), \xi \circ (\mathrm{d}\varphi_x)^{-1})$$

on T^*Q. We have

$$\begin{aligned}\Phi(TS^0 \cap T^*U) &= \varphi(U \cap S) \times \{\xi \in T^*\mathbb{R}^n \mid \xi|_{\mathbb{R}^m \times \{0\}} \equiv 0\} \\ &\cong (\mathbb{R}^m \times \{0\}) \times \Big(\{0\} \times \mathbb{R}^{n-m}\Big) \cong \mathbb{R}^n.\end{aligned}$$

Thus TS^0 is an n-dimensional submanifold of T^*Q.

Let now $(x, \xi) \in TS^0$ and $v \in T_{(x,\xi)}(TS^0)$. Let $\alpha \in \Omega^1(T^*Q)$ be the tautological 1-form. We have

$$(i^*\alpha)_{(x,\xi)}(v) = \alpha_{(x,\xi)}(v) = \xi(\mathrm{d}\pi(v)) = 0$$

because $\mathrm{d}\pi(v) \in T_xS$ and $\xi|_{T_xS} \equiv 0$. Denote by $i\colon TS^0 \hookrightarrow T^*Q$ the inclusion. The above shows that $i^*\alpha = 0$. It follows from $\omega_{\mathrm{std}} = \mathrm{d}\alpha$ that

$$i^*\omega_{\mathrm{std}} = i^*(\mathrm{d}\alpha) = \mathrm{d}(i^*\alpha) = 0.$$

Hence TS^0 is a Lagrangian submanifold of T^*Q.

10.3.

(a) Let X be a smooth manifold. Show that any diffeomorphism $\psi\colon X \to X$ lifts to a symplectomorphism

$$\Psi\colon T^*X \to T^*X$$

by the formula

$$\Psi(x, \xi) = (\psi(x), \xi \circ (\mathrm{d}\psi_x)^{-1}).$$

Solution. Let $\alpha \in \Omega^1(T^*X)$ denote the tautological 1-form. We show that $\Psi^*\alpha = \alpha$, which is even stronger than being a symplectomorphism. Let $p = (x, \xi) \in T^*X$ and $v \in T_{(x,\xi)}T^*X$. We compute

$$\begin{aligned}
(\Psi^*\alpha)_p(v) &= \alpha_{(\psi(x),\xi\circ(\mathrm{d}\psi_x)^{-1})}\left(\mathrm{d}\Psi_p(v)\right) \\
&= \left(\xi \circ (\mathrm{d}\psi_x)^{-1}\right)\left(\mathrm{d}\pi_{\Psi(p)} \circ \mathrm{d}\Psi_p(v)\right) \\
&= \left(\xi \circ (\mathrm{d}\psi_x)^{-1}\right)\left(\mathrm{d}(\pi \circ \Psi)_p(v)\right) \\
&= \left(\xi \circ (\mathrm{d}\psi_x)^{-1}\right)\left(\mathrm{d}(\psi \circ \pi)_p(v)\right) \\
&= \left(\xi \circ (\mathrm{d}\psi_x)^{-1} \circ \mathrm{d}\psi_x\right)\left(\mathrm{d}\pi_p(v)\right) \\
&= \xi(\mathrm{d}\pi_p(v)) \\
&= \alpha_p(v),
\end{aligned}$$

hence $\Psi^*\alpha = \alpha$ as claimed.

(b) Consider $(\mathbb{C}^2 \cong \mathbb{R}^4, \omega_{\mathrm{std}})$ and the submanifold

$$L_{\mathrm{Ch}} := \left\{ \begin{pmatrix} (e^s + ie^{-s}t)\cos(\theta) \\ (e^s + ie^{-s}t)\sin(\theta) \end{pmatrix} \middle|\, \theta, s, t \in \mathbb{R}, s^2 + t^2 = 1 \right\}.$$

Show that this is a Lagrangian submanifold. It is called *Chekanov torus.*

Hint: Write L_{Ch} as the image of a Lagrangian in $T^*S^1 \times T^*\mathbb{R}$ under a symplectomorphism.

Solution. Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ the coordinate on the unit circle S^1 and consider the diffeomorphism

$$\begin{aligned}\psi\colon S^1 \times \mathbb{R} &\to \mathbb{R}^2\backslash\{0\},\\ (\theta, s) &\mapsto (e^s\cos(\theta), e^s\sin(\theta)).\end{aligned}$$

We lift it to a symplectomorphism $\Psi\colon T^*(S^1 \times \mathbb{R}) \to T^*(\mathbb{R}^2\backslash\{0\})$ as in part (a). We identify $T^*(S^1\times\mathbb{R}) \cong T^*S^1 \times T^*\mathbb{R}$ and denote elements in it by quadruples (θ, τ, s, t), where $(\theta, \tau) \in T^*S^1 \cong S^1 \times \mathbb{R}$ and $(s, t) \in T^*\mathbb{R} \cong \mathbb{R}^2$. Furthermore we identify $T^*(\mathbb{R}^2\backslash\{0\}) \cong (\mathbb{R}^2\backslash\{0\}) \times \mathbb{R}^2 \subset \mathbb{R}^4 \cong \mathbb{C}^2$.

In these coordinates we have

$$\mathrm{d}\psi_{(\theta,s)} = \begin{pmatrix} -e^s\sin(\theta) & e^s\cos(\theta) \\ e^s\cos(\theta) & e^s\sin(\theta)\end{pmatrix}$$

and thus

$$(\mathrm{d}\psi_{(\theta,s)})^{-1} = \begin{pmatrix} -e^{-s}\sin(\theta) & e^{-s}\cos(\theta) \\ e^{-s}\cos(\theta) & e^{-s}\sin(\theta)\end{pmatrix}.$$

It follows that for $\xi = (\tau, t) \in T^*_{(\theta,s)}(S^1 \times \mathbb{R}) \cong \mathrm{Hom}(\mathbb{R}^2, \mathbb{R})$ we have

$$\xi \circ (\mathrm{d}\psi_{(\theta,s)})^{-1} = \left(-e^{-s}\sin(\theta)\tau + e^{-s}\cos(\theta)t,\ e^{-s}\cos(\theta)\tau + e^{-s}\sin(\theta)t\right).$$

As noted above, we identify $T^*(\mathbb{R}^2 \setminus \{0\} \cong (\mathbb{R}^2 \setminus \{0\}) \times \mathbb{R}^2$ and consider it as a subset of $\mathbb{C}^2$. If we think of the base as living in the real part of $\mathbb{C}^2$ and the fibres in the purely imaginary part, then Ψ is given by

$$\Psi(\theta, \tau, s, t) = \begin{pmatrix} (e^s + ie^{-s}t)\cos(\theta) - i\tau e^{-s}\sin(\theta) \\ (e^s + ie^{-s}t)\sin(\theta) + i\tau e^{-s}\cos(\theta)\end{pmatrix}.$$

The Chekanov torus L_{Ch} is the image under Ψ of the product of the zero section in T^*S^1 with the unit circle in $\mathbb{R}^2$. Hence it is the image of a Lagrangian under a symplectomorphism. We conclude that L_{Ch} is a Lagrangian.