The most important exercises are marked with an asterisk *.

***11.1.** Let (M, ω) be a closed symplectic manifold.

(a) Let X be a symplectic vector field, i.e. a vector field satisfying $\mathrm{d}\iota_X\omega = 0$. Let $\psi \in \mathrm{Symp}_0(M, \omega)$. Show that $\psi^*X - X$ is a Hamiltonian vector field.

Solution. Let ψ_t be a symplectic isotopy connecting id to ψ and let Y_t be the generating vector field. It is a symplectic vector field for every t. We compute

$$\begin{aligned}
\iota_{(\psi^*X-X)}\omega &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left(\iota_{\psi_t^*X}\omega\right)\mathrm{d}t \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t^*\iota_X\omega\right)\mathrm{d}t \\
&= \int_0^1 \mathcal{L}_{Y_t}(\iota_X\omega)\,\mathrm{d}t \\
&= \int_0^1 \left(\mathrm{d}\iota_{Y_t}\iota_X\omega + \iota_{Y_t}\mathrm{d}\iota_X\omega\right)\mathrm{d}t \\
&= \int_0^1 \mathrm{d}\iota_{Y_t}(\iota_X\omega)\,\mathrm{d}t \\
&= \int_0^1 \mathrm{d}\omega(X, Y_t)\,\mathrm{d}t \\
&= \mathrm{d}\left(\int_0^1 \omega(X, Y_t)\,\mathrm{d}t\right).
\end{aligned}$$

This shows that $\psi^*X - X$ is a Hamiltonian vector field.

(b) Suppose the smooth isotopies $\varphi_t, \psi_t \in \mathrm{Diff}(M)$ are generated by the vector fields X_t and Y_t. Compute the vector field that generates $\psi_t \circ \varphi_t$.

Solution. As in Exercise 2.2, we compute

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\psi_t \circ \varphi_t\right) &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\psi_t\right) \circ \varphi_t + \mathrm{d}\psi_t\left(\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t\right) \\
&= Y_t \circ \psi_t \circ \varphi_t + \mathrm{d}\psi_t\left(X_t \circ \varphi_t\right) \\
&= \left(Y_t + \mathrm{d}\psi_t(X_t \circ \psi_t^{-1})\right) \circ \left(\psi_t \circ \varphi_t\right) \\
&= \left(Y_t + \left(\psi_t^{-1}\right)^* X_t\right)\left(\psi_t \circ \varphi_t\right).
\end{aligned}$$

Therefore, $\psi_t \circ \varphi_t$ is generated by $Y_t + \left(\psi_t^{-1}\right)^* X_t$.

(c) Deduce that

$$\mathrm{Flux}\colon \widetilde{\mathrm{Symp}_0}(M, \omega) \to \mathrm{H}^1(M, \mathbb{R})$$

is a homomorphism of groups.

Solution. Let $\{\varphi_t\}$ and $\{\psi_t\}$ be symplectic isotopies representing elements in $\widetilde{\text{Symp}}_0(M,\omega)$. Let X_t and Y_t be the vector fields generating them. We compute

$$\begin{aligned}\text{Flux}(\{\psi_t\circ\varphi_t\}) &\overset{(b)}{=} \int_0^1 [\iota_{Y_t+(\psi_t^{-1})^*(X_t)}\omega]\mathrm{d}t\\ &= \int_0^1 [\iota_{Y_t}\omega] + [\iota_{(\psi_t^{-1})^*(X_t)}\omega]\mathrm{d}t\\ &\overset{(a)}{=} \int_0^1 [\iota_{Y_t}\omega] + [\iota_{X_t}\omega]\mathrm{d}t\\ &= \text{Flux}(\{\psi_t\}) + \text{Flux}(\{\varphi_t\}).\end{aligned}$$

In the equality labelled by (b) we used part (b) of this exercise and in the equality (a) we used the following: applying part (a) of this exercise to ψ_t^{-1} and the symplectic vector field X_t we get that

$$\iota_{(\psi_t^{-1})^*(X_t)}\omega - \iota_{X_t}\omega = \iota_{(\psi_t^{-1})^*(X_t)-X_t}\omega$$

is exact. In particular,

$$[\iota_{(\psi_t^{-1})^*(X_t)}\omega] = [\iota_{X_t}\omega] \in \mathrm{H}^1(M,\mathbb{R}).$$

***11.2.** Let (M,ω) be a closed symplectic manifold. Show that Flux is surjective.

Solution. Let $\theta\in\Omega^1(M,\mathbb{R})$ be a closed 1-form representing an element $[\theta]$ in $H^1(M,\mathbb{R})$. We show that there exists a symplectic isotopy $\{\psi_t\}_{t\in[0,1]}$ starting at $\psi_0=\text{id}$ such that $\text{Flux}(\{\psi_t\})=[\theta]$. Let X be the smooth vector field defined by $\iota_X\omega=\theta$. (This is well-defined because ω is non-degenerate.) The vector field X is a symplectic vector field, since θ is closed. Let ψ_t, $t\in[0,1]$, be the flow generated by X. Since X is symplectic, ψ_t is a symplectic isotopy. Moreover,

$$\text{Flux}(\{\psi_t\}) = \int_0^1 [\iota_X\omega]\mathrm{d}t = \int_0^1 [\theta]\mathrm{d}t = [\theta],$$

which proves the result.

11.3. Let (M,ω) be a closed symplectic manifold. Let ψ_t and φ_t be symplectic isotopies with $\psi_0=\varphi_0=\text{id}$. Consider the *juxtaposition* of ψ_t and φ_t, which is defined by

$$\chi_t = \begin{cases}\varphi_{2t}, & 0\le t\le \frac{1}{2},\\ \psi_{2t-1}\circ\varphi_1, & \frac{1}{2}\le t\le 1.\end{cases}$$

Prove that χ_t and $\psi_t \circ \varphi_t$ represent the same element in $\widetilde{\mathrm{Symp}_0}(M,\omega)$. Use it to give a different proof for Flux being a homomorphism.

Solution. We have to show that χ_t and $\psi_t \circ \varphi_t$ are homotopic with fixed endpoints. For this, consider the homotopy

$$f_{s,t} := \begin{cases} \psi_0 \circ \varphi_{\frac{2}{2-s}t} & 0 \leq t \leq \frac{s}{2} \\ \psi_{\frac{2}{2-s}t + \frac{s}{s-2}} \circ \varphi_{\frac{2}{2-s}t} & \frac{s}{2} \leq t \leq 1 - \frac{s}{2} \\ \psi_{\frac{2}{2-s}t + \frac{s}{s-2}} \circ \varphi_1 & 1 - \frac{s}{2} \leq t \leq 1 \end{cases}$$

We have $f_{0,t} = \psi_t \circ \varphi_t$, $f_{1,t} = \chi_t$, $f_{s,0} = \mathrm{id}$ and $f_{s,1} = \psi_1 \circ \varphi_1$. Therefore, $f_{s,t}$ is the required homotopy between $\psi_t \circ \varphi_t$ and χ_t.

Since the Flux is well-defined on $\widetilde{\mathrm{Symp}_0}(M,\omega)$ we can now use juxtaposition of paths to represent composition and compute its flux using the generating vector field. Let X_t and Y_t be the generating vector fields for $\{\varphi_t\}$ and $\{\psi_t\}$. Then the generating vector field for the juxtaposition χ_t is

$$\begin{cases} 2X_{2t} & 0 \leq t \leq \frac{1}{2} \\ 2Y_{2t-1} & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Thus

$$\begin{aligned} \mathrm{Flux}(\{\psi_t \circ \varphi_t\}) &= \mathrm{Flux}(\{\chi_t\}) \\ &= \int_0^{\frac{1}{2}} [\iota_{2X_t}\omega]\,\mathrm{d}t + \int_{\frac{1}{2}}^1 [\iota_{2Y_{2t-1}}\omega]\,\mathrm{d}t \\ &= \int_0^{\frac{1}{2}} 2[\iota_{X_t}\omega]\,\mathrm{d}t + \int_{\frac{1}{2}}^1 2[\iota_{Y_{2t-1}}\omega]\,\mathrm{d}t \\ &= \int_0^1 [\iota_{X_t}\omega]\,\mathrm{d}t + \int_0^1 [\iota_{Y_t}\omega]\,\mathrm{d}t \\ &= \mathrm{Flux}(\{\varphi_t\}) + \mathrm{Flux}(\{\psi_t\}), \end{aligned}$$

where we used the substitutions $t \mapsto 2t$ and $t \mapsto 2t-1$ in the second last step.