The most important exercises are marked with an asterisk *.

12.1. Let (M, ω) be a closed symplectic manifold. Consider

$$M \times M^- := \Big(M \times M,\ \omega \oplus (-\omega)\Big).$$

Let $j_\Delta \colon M \to M \times M^-$ denote the diagonal inclusion $j(x) = (x, x)$.

Let ψ_t, $t \in [0,1]$, be a symplectic isotopy with $\psi_0 = \mathrm{id}$. Show that

$$\operatorname{Flux}(\{\psi_t\}) = -j_\Delta^* \operatorname{Flux}(\{\mathrm{id} \times \psi_t\}).$$

Solution. Let X_t be the vector field generating ψ_t. Then the symplectic isotopy $\mathrm{id} \times \psi_t$ is generated by the vector field

$$\hat{X}_t(x, y) = (0, X_t(y)) \in T_{(x,y)}(M \times M^-).$$

Let $x \in M$ and $v \in T_x M$. We compute

$$\begin{aligned} \Big(j_\Delta^* \iota_{\hat{X}_t}(\omega \oplus (-\omega))\Big)_x (v) &= (\omega \oplus (-\omega))(\hat{X}_t(x,x), (v,v)) \\ &= (-\omega)(X_t(x), v) \\ &= -\left(\iota_{X_t}\omega\right)_x (v). \end{aligned}$$

Integrating over $t \in [0,1]$ and taking classes in $H^1(M, \mathbb{R})$ yields the result.

***12.2.** Let $\chi \colon M \to M$ be a symplectomorphism on a closed symplectic manifold (M, ω) and let ψ_t, $t \in [0,1]$, be a symplectic isotopy with $\psi_0 = \mathrm{id}$. Show that

$$\operatorname{Flux}(\{\chi^{-1} \circ \psi_t \circ \chi\}) = \chi^*(\operatorname{Flux}(\{\psi_t\}).$$

Solution. Let X_t be the vector field generating ψ_t. Then the symplectic isotopy $\chi^{-1} \circ \psi_t \circ \chi$ is generated by the vector field

$$\chi^*(X_t) = \mathrm{d}\chi^{-1}(X_t \circ \chi)$$

(see Exercise 2.2.) Thus for $x \in M$ and $v \in T_x M$ we have

$$\begin{aligned} \Big(\iota_{\chi^*(X_t)}\omega\Big)_x (v) &= \omega(\mathrm{d}\chi^{-1}(X_t \circ \chi(x)), v) \\ &= \omega(X_t \circ \chi(x), \mathrm{d}\chi(v)) \\ &= \left(\iota_{X_t}\omega\right)_{\chi(x)} (\mathrm{d}\chi(v)) \\ &= \left(\chi^*(\iota_{X_t}\omega)\right)_x (v), \end{aligned}$$

thus $\iota_{\chi^*(X_t)}\omega = \chi^*(\iota_{X_t}\omega)$. Therefore,

$$\begin{aligned}\operatorname{Flux}(\{\chi^{-1}\circ\psi_t\circ\chi\}) &= \int_0^1 [\iota_{\chi^*(X_t)}\omega]\mathrm{d}t\\ &= \int_0^1 [\chi^*(\iota_{X_t}\omega)]\mathrm{d}t\\ &= \chi^*\left(\int_0^1 [\iota_{X_t}\omega]\mathrm{d}t\right)\\ &= \chi^*\operatorname{Flux}(\{\psi_t\}).\end{aligned}$$

***12.3.** Consider the exact symplectic manifold $(T^*Q, \mathrm{d}\alpha)$, where $\alpha \in \Omega^1(Q)$ is the canonical 1-form on M. If $\sigma \in \Omega^1(Q)$ is a closed 1-form, there is an associated diffeomorphism $\nu_\sigma\colon T^*Q \to T^*Q$ defined by

$$\nu_\sigma(q,\xi) = (q, \xi + \sigma_q).$$

(a) Prove that

$$\nu_\sigma^*\alpha - \alpha = \pi^*\sigma,$$

where $\pi\colon T^*Q \to Q$ denotes the canonical projection.

Solution.

Let $q \in Q$, $\xi \in T_q^*Q$ and $v \in T_{(q,\xi)}Q$. We compute

$$\begin{aligned}(\nu_\sigma^*\alpha - \alpha)_{(q,\xi)}(v) &= \alpha_{(q,\xi+\sigma_q)}(\mathrm{d}\nu_\sigma(v)) - \alpha_{(q,\xi)}(v)\\ &= (\xi+\sigma_q)(\mathrm{d}(\pi\circ\nu_\sigma)(v)) - \xi(\mathrm{d}\pi(v))\\ &= (\xi+\sigma_q)(\mathrm{d}\pi(v)) - \xi(\mathrm{d}\pi(v))\\ &= \sigma_q(\mathrm{d}\pi(v))\\ &= (\pi^*\sigma)_{(q,\xi)}(v).\end{aligned}$$

This shows the claim.

(b) Prove that ν_σ is a symplectomorphism.

Solution. Since σ is assumed to be closed, $\nu_\sigma^*\alpha - \alpha = \pi^*\sigma$ is closed. Therefore, ν_σ is a symplectomorphism (Proposition Lecture 21).

(c) Prove that ν_σ is a Hamiltonian diffeomorphism if and only if σ is exact.

Solution. By the Proposition from Lecture 21 and part (a), ν_σ is a Hamiltonian diffeomorphism if and only if $\pi^*\sigma$ is exact. This is equivalent to σ being exact. (Indeed, if $\sigma = \mathrm{d}f$ is exact, then $\pi^*\sigma = \pi^*\mathrm{d}f = \mathrm{d}\pi^*f$ is exact. Conversely, if $\pi^*\sigma = \mathrm{d}F$ is exact, then $\sigma = (\pi\circ j)^*\sigma = j^*\pi^*\sigma = j^*\mathrm{d}F = \mathrm{d}j^*F$ is exact, where $j\colon Q \to T^*Q$ denotes the inclusion of the zero-section.)