This is a sample exam question.

*13.1. Hamiltonian diffeomorphisms

Let (M, ω) be a closed symplectic manifold.

(a) (3 points) What is a Hamiltonian diffeomorphism on M?

Solution. Let $H\colon [0,1] \times M \to \mathbb{R}$ be a smooth function and write $H_t = H(t, -)$ for $t \in [0,1]$. The associated time-dependent Hamiltonian vector field X_t^H on M is defined via

$$\iota_{X_t^H}\omega = -\mathrm{d}H_t$$

for $t \in [0,1]$. Its flow $\psi_t^H \in \mathrm{Diff}(M)$, $t \in [0,1]$, is defined via

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^H = X_t^H \circ \psi_t^H, \quad \psi_0^H = \mathrm{id}\,.$$

A diffeomorphism φ on M is called *Hamiltonian diffeomorphism* if it is the time-1 map $\varphi = \psi_1^H$ for some Hamiltonian function H as above. (Equivalently if it is the time-t map $\varphi = \psi_t^H$ for some H and t.)
1 point for the definition of X_t^H.
1 point for the flow ψ_t^H.
1 point for saying that ψ_1^H are the Hamiltonian diffeomorphisms.

(b) (4 points) Let $H\colon [0,1] \times M \to \mathbb{R}$ be a smooth Hamiltonian function and ψ_t^H the corresponding Hamiltonian flow. Let χ be a symplectomorphism on M. Show that $\chi^{-1}\psi_t^H\chi$ is generated by $H_t \circ \chi$.

Solution. We compute

$$\begin{aligned}\mathrm{d}(H_t \circ \chi) &= \mathrm{d}H_t \circ \mathrm{d}\chi \\ &= -\omega\left(X_t^H \circ \chi, \mathrm{d}\chi(-)\right) \\ &= -\omega\left(\mathrm{d}\chi^{-1}\left(X_t^H \circ \chi\right), -\right),\end{aligned}$$

hence $X_t^{H\circ\chi} = \chi^*(X_t^H)$. Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\chi^{-1}\psi_t^H\chi\right) &= \mathrm{d}\chi^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^H \circ \chi\right) \\ &= \chi^*\left(\frac{\mathrm{d}}{\mathrm{d}t}\psi_t^H\right) = \chi^*(X_t^H \circ \psi_t^H) = X_t^{H\circ\chi} \circ \left(\chi^{-1}\psi_t^H\chi\right),\end{aligned}$$

which proves the claim.
2 points for expressing $X_t^{H\circ\chi}$ in terms of X_t^H.
2 points for showing that $X_t^{H\circ\chi}$ generates $\chi^{-1}\psi_t^H\chi$.

(c) (4 points) Consider the 2-sphere $S^2 \subset \mathbb{R}^3$ endowed with the standard symplectic form given by

$$\omega_x(v, w) = x \cdot (v \times w),$$

for all $x \in S^2$ and $v, w \in T_x S^2 = \{v \in \mathbb{R}^3 \mid x \cdot v = 0\}$. Let $H\colon S^2 \to \mathbb{R}$ be the autonomous Hamiltonian function given by

$$(x_1, x_2, x_3) \mapsto x_3.$$

Compute the corresponding Hamiltonian flow ψ_t^H, $t \in \mathbb{R}$.

Solution. For $x \in S^2$ and $v \in T_x S^2$ the equation for the Hamiltonian vector field is

$$x \cdot (X^H(x) \times v) = \omega_x(X^H(x), v) = -\mathrm{d}H_x(v) = -v_3$$

for $v \in T_x S^2$. Writing $X^H(x) = \left(X_1^H, X_2^H, X_3^H\right)$ this equation becomes

$$x_1(X_2^H v_3 - X_3^H v_2) + x_2(X_3^H v_1 - X_1^H v_3) + x_3(X_1^H v_2 - X_2^H v_1) = -v_3. \quad (1)$$

In trying to solve this equation it's helpful to have a good guess: Since H is autonomous, X^H should point along circles in S^2 with $H(x) = x_3 = \mathit{const}$. Therefore $X^H(x)$ should be parallel to $(x_2, -x_1, 0)$. Plugging in this guess into equation (1) and using $x \cdot v = 0$ and $x_1^2 + x_2^2 + x_3^2 = 1$ we see that

$$X^H(x) = \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}$$

actually solves the equation. To get the flow, we need to solve:

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t = X^H \circ \psi_t.$$

If we denote $\psi_t = (\psi_t^1, \psi_t^2, \psi_t^3)$, we immediately see that $\psi_t^3(x) = x_3$. To get ψ_t^1 and ψ_t^2, we solve the system:

$$\dot{\psi}_t^1 = x_2 \circ \psi_t = \psi_t^2$$
$$\dot{\psi}_t^2 = -x_1 \circ \psi_t = -\psi_t^1.$$

Putting everything together, we obtain:

$$\psi_t^H(x) = (\cos(t)x_1 + \sin(t)x_2,\ -\sin(t)x_1 + \cos(t)x_2,\ x_3)$$

This is rotation along the x_3-axis.
1 point for explicit equation for X^H.
1 point for realising that X^H is tangent to level sets.
1 point for X^H.
1 point for ψ_t^H.