Week 1: σ-fields and measures

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 25/09/2023 17:00 (online) following the instructions on the course website:

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

***

1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. **($\liminf$ and $\limsup$ of sets)**
Let (A_n) be a sequence of events on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Recall the definitions $\liminf_{n\to\infty} A_n = \cup_{n\geq 1} \cap_{m\geq n} A_m$ and $\limsup_{n\to\infty} A_n = \cap_{n\geq 1} \cup_{m\geq n} A_m$.

(1) If $A = \mathbb{R}$, give the set $\limsup_{n\to\infty} A_n$ in the following three cases (please justify your answers):
  (a) $A_n = [-1/n, 3+1/n]$
  (b) $A_n = [-2-(-1)^n, 2+(-1)^{n+1})$
  (c) $A_n = p_n\mathbb{N}$, where $(p_n)_{n\geq 1}$ is the sequence of prime numbers and $p_n\mathbb{N}$ denotes the set of all multiples of p_n.

(2) Show that

$$\mathbb{P}\left(\liminf_{n\to\infty} A_n\right) \leq \liminf_{n\to\infty} \mathbb{P}(A_n) \leq \limsup_{n\to\infty} \mathbb{P}(A_n) \leq \mathbb{P}\left(\limsup_{n\to\infty} A_n\right).$$

2 Training exercises

Exercise 2. **(0 or 1)**
Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space.

(1) Let A and B two events with $A \subset B$. Assume that $\mathbb{P}(A) = 1$; what can be said of $\mathbb{P}(B)$? And if $\mathbb{P}(B) = 0$, what can be said of $\mathbb{P}(A)$?

Let $(A_i)_{i\geq 1}$ be a sequence of events.

(2) Assume that $\mathbb{P}(A_i) = 0$ for every $i \geq 1$. Show that $\mathbb{P}\left(\cap_{i=1}^{\infty} A_i\right) = 0$ and that $\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) = 0$.

(3) Assume that $\mathbb{P}(A_i) = 1$ for every $i \geq 1$. Show that $\mathbb{P}\left(\cup_{i=1}^{\infty} A_i\right) = 1$ and that $\mathbb{P}\left(\cap_{i=1}^{\infty} A_i\right) = 1$.

Exercise 3. **(Generating π-systems)** A collection of sets $\mathcal{A}$ is said to be a generating π-system of a σ-field $\mathcal{B}$ is $\sigma(\mathcal{A}) = \mathcal{B}$ and if $\mathcal{A}$ is closed under finite intersections (meaning that for every $A, B \in \mathcal{A}$ we have $A \cap B \in \mathcal{A}$.

(1) Show that $\mathcal{A} = \{[0,a] : a \in [0,1]\}$ is a π-system generating $\mathcal{B}([0,1])$.

(2) Prove that $\mathcal{A}' = \{(-\infty, a_1] \times \cdots \times (-\infty, a_d] : a_1, \ldots, a_d \in \mathbb{R}\} \cup \{\mathbb{R}^d\}$ is a π-system generating the σ-algebra $\mathcal{B}(\mathbb{R}^d)$.

Exercise 4. **(Questions and operations on σ-fields)**

(1) Is the set of all open sets of $\mathbb{R}$ a σ-field?

(2) For every $n \geq 0$, define on $\mathbb{N}$ the σ-field $\mathcal{F}_n = \sigma(\{0\}, \{1\}, \ldots, \{n\})$. Show that the sequence of σ-fields $(\mathcal{F}_n, n \geq 0)$ is non-decreasing but that $\bigcup_{n \geq 0} \mathcal{F}_n$ is not a σ-field.
Hint: argue by contradiction and use the subset of even integers.

(3) We throw two coins. To model the outcome, we use the probability space $\Omega = \{00, 01, 10, 11\}$ equiped with the σ-field $\mathcal{P}(\Omega)$. Let $\mathbb{P}$ be the probability measure on Ω corresponding to the case where the two coins are fair and are thrown independently. Let $\mathbb{Q}$ be the probability measure on Ω corresponding to the case where the second coin is rigged and always gives the same result as the first one. Show that the set $\{A \in \mathcal{P}(\Omega) : \mathbb{P}(A) = \mathbb{Q}(A)\}$ is not a σ-field (this gives in particular an example of a Dynkin system which is not a σ-field).

(4) Let $(E \times F, \mathcal{A})$ be a measured space and $\pi : E \times F \longrightarrow E$ the canonical projection defined by $\pi(x,y) = x$. Is the set $\mathcal{A}_E := \{\pi(A), A \in \mathcal{A}\}$ always a σ-field?

(5) Let $(E, \mathcal{A})$ be a measurable space. Let $\mathcal{C}$ be a collection of subsets of E, and fix $B \in \sigma(\mathcal{C})$. Alexandra says: there always exists a countable collection $\mathcal{D} \subset \mathcal{C}$ such that $B \in \sigma(\mathcal{D})$. Is she correct?

3 More involved exercise (optional, will not be coverd in the exercise class)

Exercise 5. Let $(E, \mathcal{E}, \mu)$ be a measured space with μ finite. Let $\mathcal{A}$ be a collection of subsets such that:
(a) $E \in \mathcal{A}$ (b) if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$ (c) if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$ (d) $\sigma(\mathcal{A}) = \mathcal{E}$.

The goal of this exercise is to show that for every $E \in \mathcal{E}$, for every $\epsilon > 0$ there exists $A \in \mathcal{A}$ such that $\mu(E \Delta A) \leq \epsilon$ (where $X \Delta Y = (X \cup Y) \backslash (X \cap Y)$). To this end, set $\mathcal{S} = \{E \in \mathcal{E} : \forall \epsilon > 0, \exists A \in \mathcal{A} : \mu(E \Delta A) \leq \epsilon\}$.

(1) Show that $\mathcal{S}$ is stable by finite unions.

(2) Show that $\mathcal{S}$ is a σ-field (for stability by countable unions, justify that one may assume that the events are pairwise disjoint).

(3) (*Application to percolation*)

(a) Set $E = \{0,1\}^{\mathbb{Z}^2}$. We see the elements of E as $\mathbb{Z}^2$, with each vertex either occupied (value 1), or inoccupied (value 0). Denote by $B_n = \{x \in \mathbb{Z}^2 : \|x\|_\infty \leq n\}$ the "box" of size n. We say that $A \subset E$ is a cylinder of size n if there exists $(u_s) \in \{0,1\}^{B_n}$ such that

$$A = \{(x_s) \in E : x_s = u_s \, \forall s \in B_n\}.$$

Denote by C_n the set of cylinders of size n. We then take

$$\mathcal{E} = \sigma\left(\bigcup_{n \geq 1} C_n\right).$$

Connect two vertices at distance 1 with an edge if they are both occupied. Justify that the event "there exists an infinite path between connected vertices" is measurable (that it, in $\mathcal{E}$).

(b) Every vertex of $\mathbb{Z}^2$ is occupied independently with probability $p \in [0,1]$. For $x \in \mathbb{Z}^2$, denote by τ_x the translation by vector x. If $A \subset \{0,1\}^{\mathbb{Z}^2}$ is an event, we write $\tau_x A = \{(u_s) \in \{0,1\}^{\mathbb{Z}^2} : \tau_x^{-1}(u_s) \in A\}$. Let A be an event invariant under translations (that is $\tau_x A = A$ for every $x \in \mathbb{Z}^2$). Show that $\mathbb{P}(A)$ is 0 or 1. In particular, the probability of having an infinite connected component (in the sense of question (a)) is 0 or 1.

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 6. (The duelling idiots, taken from: *Duelling idiots and other probability puzzlers* P.J.Nahin, Princeton Univ.Press (2000)).

A and B decide to duel but they have just one gun (a six shot revolver) and only one bullet. Being dumb, this does not deter them and they agree to "duel" as follows: They will insert the lone bullet into the gun's cylinder, A will then spin the cylinder and shoot at B (who, standing inches away, is impossible to miss). If the gun doesn't fire then A will give the gun to B, who will spin the cylinder and then shoot at A. This back and forth duel will continue until one fool shoots the other.

What is the probability that A will win?