Week 10: Uniformly integrable martingales, stopping times

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 27/11/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let $(M_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$ martingale and let T be a $(\mathcal{F}_n)_{n\geq 0}$ stopping time.

(1) Assume that T is bounded. Show that $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

(2) Assume that $\mathbb{E}[T] < \infty$ and there there exists $K > 0$ such that a.s. we have $\mathbb{E}[|M_{n+1} - M_n| \mid \mathcal{F}_n] \leq K$ for every $n \geq 0$. Show that $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

Hint. Justify that $|M_{T\wedge n}| \leq |M_0| + \sum_{i=0}^{\infty} |M_{i+1} - M_i| \mathbb{1}_{T>i}$ and use dominated convergence.

(3) Let $(X_n)_{n\geq 1}$ be i.i.d. integrable real-valued random variables. Set $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n \geq 1$ and $\mathcal{F}_n = \sigma(S_i : 0 \leq i \leq n)$ for $n \geq 0$. Finally, let T be a $(\mathcal{F}_n)$-stopping time with $\mathbb{E}[T] < \infty$. Show that

$$\mathbb{E}[S_T] = \mathbb{E}[X_1]\mathbb{E}[T].$$

2 Training exercises

Exercise 2. Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration and let S, T be two stopping times with respect to $(\mathcal{F}_n)_{n\geq 0}$. Let $S, T : \Omega \to \mathbb{N} \cup \{\infty\}$ be $(\mathcal{F}_n)$ stopping times. Prove or disprove with a counter-example the following statements:

(1) $S \vee T$ is a stopping time.

(2) $S \wedge T$ is a stopping time.

(3) $S + T$ is a stopping time.

(4) $S + 1$ is a stopping time.

(5) $S - 1$ is a stopping time.

Exercise 3. Let $(\mathcal{F}_n)$ be a filtration and $\mathcal{F}_\infty := \sigma(\cup_n \mathcal{F}_n)$. Let (X_n) be a sequence of integrable random variables such that $X_n \to X$ as $n \to \infty$ both a.s. and in L^1, where X is an integrable random variable. Assume that for all $n \geq 0$, $|X_n| \leq Y$ a.s., where Y is a non-negative integrable random variable.

(1) Define $Z_n = \sup_{m \geq n} |X_m - X|$. Show that $Z_n \to 0$ as $n \to \infty$ a.s. and in L^1.

(2) Show that $\mathbb{E}[X_n \mid \mathcal{F}_n] \to \mathbb{E}[X \mid \mathcal{F}_\infty]$ as $n \to \infty$ a.s.

(3) Let (Y_n) and (Z_n) be two independent sequences of independent random variables such that $\mathbb{P}(Y_n = n) = n^{-2} = 1 - \mathbb{P}(Y_n = 0)$ and $\mathbb{P}(Z_n = n) = n^{-1} = 1 - \mathbb{P}(Z_n = 0)$. Set $X_n = Y_n Z_n$ and $\mathcal{A} = \sigma(Z_n : n \geq 0)$.

Show that $X_n \to 0$ as $n \to \infty$ both a.s. and in L^1, but $\mathbb{E}[X_n \mid \mathcal{A}]$ does not converge to 0 almost surely.

Exercise 4. Let T be a stopping time for a filtration $(\mathcal{F}_n)_{n \geq 0}$. Assume that there exit $\varepsilon > 0$ and $n_0 \geq 1$ such that for every $n \geq 0$, almost surely

$$\mathbb{P}(T \leq n + n_0 | \mathcal{F}_n) > \varepsilon.$$

(1) Show that for every $k \geq 0$ we have $\mathbb{P}(T \geq k n_0) \leq (1-\varepsilon)^k$.

(2) Show that T is almost surely finite and that $\mathbb{E}[T] < \infty$.

Exercise 5. Let $(M_n)_{n \geq 0}$ be a uniformly integrable martingale with respect to a filtration $(\mathcal{F}_n)_{n \geq 0}$.

(1) Is it true that the collection $\{M_T : T \text{ stopping time with respect to } (\mathcal{F}_n)_{n \geq 0}\}$ is uniformly integrable?

(2) Let T be a stopping time. Is it true that $(M_{n \wedge T})_{n \geq 0}$ is a uniformly integrable martingale? Justify your answer.

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 6. Let T be a stopping time with respect to a filtration $(\mathcal{F}_n)_{n \geq 0}$ with $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$. Recall that the σ-field $\mathcal{F}_T$ is defined by

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T = n\} \in \mathcal{F}_n \ \forall n \geq 0\}.$$

(1) Let S be a stopping time with respect to the filtration $(\mathcal{F}_n)_{n \geq 0}$ such that $S \leq T$. Show that $\mathcal{F}_S \subset \mathcal{F}_T$.

(2) Show that T is $\mathcal{F}_T$ measurable.

(3) Here we assume that $(X_n)_{n \geq 0}$ is a sequence of random variables and that $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. Show that $\mathcal{F}_T = \sigma(X_{T \wedge n} : n \geq 0)$.

Exercise 7. Let (M_n) be a martingale with respect to a filtration $(\mathcal{F}_n)$ and let S and T be stopping times. Show that for every $n \geq 0$ we almost surely have

$$\mathbb{E}(M_{n\wedge S} \mid \mathcal{F}_T) = M_{n\wedge S\wedge T}$$

Exercise 8. Let $f : [0,1] \to \mathbb{R}$ be a Lipschitz function, i.e. there exists $K > 0$ such that $|f(x) - f(y)| \leq K|x-y|$ for all $x, y \in [0,1]$. Let $f_n' : [0,1] \to \mathbb{R}$ be defined by

$$f_n'(x) = \begin{cases} 2^n \left(f\left(\frac{i+1}{2^n}\right) - f\left(\frac{i}{2^n}\right)\right) & : x \in [i/2^n, (i+1)/2^n)\,, \\ 0 & : x = 1\,. \end{cases}$$

Note that f_n' is the derivative of the piecewise linear extension of $f|_{(2^{-n}\mathbb{Z})\cap[0,1]}$ to $[0,1]$.

(1) Show that $f_n' \to f'$ almost everywhere and in L^1 (with respect to the Lebesgue measure) as $n \to \infty$ for some integrable function $f' : [0,1] \to \mathbb{R}$.

Hint. Use the martingale convergence theorem after defining a suitable probability space together with a martingale on it.

(2) Deduce that

$$f(x) - f(0) = \int_0^x f'(y)\,dy \quad \text{for all } x \in [0,1]\,.$$

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 9. You throw a fair six-sided die until you get 6. What is the expected number of throws (including the throw giving 6) conditioned on the event that all throws gave even numbers?