Week 10: Uniformly integrable martingales, stopping times

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 27/11/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let $(M_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$ martingale and let T be a $(\mathcal{F}_n)_{n\geq 0}$ stopping time.

(1) Assume that T is bounded. Show that $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

(2) Assume that $\mathbb{E}[T] < \infty$ and there there exists $K > 0$ such that a.s. we have $\mathbb{E}[|M_{n+1} - M_n| \mid \mathcal{F}_n] \leq K$ for every $n \geq 0$. Show that $\mathbb{E}[M_T] = \mathbb{E}[M_0]$.

Hint. Justify that $|M_{T\wedge n}| \leq |M_0| + \sum_{i=0}^{\infty} |M_{i+1} - M_i| \mathbb{1}_{T>i}$ and use dominated convergence.

(3) Let $(X_n)_{n\geq 1}$ be i.i.d. integrable real-valued random variables. Set $S_0 = 0$, $S_n = X_1 + \cdots + X_n$ for $n \geq 1$ and $\mathcal{F}_n = \sigma(S_i : 0 \leq i \leq n)$ for $n \geq 0$. Finally, let T be a $(\mathcal{F}_n)$-stopping time with $\mathbb{E}[T] < \infty$. Show that

$$\mathbb{E}[S_T] = \mathbb{E}[X_1]\mathbb{E}[T].$$

Solution:

(1) Let $N > 0$ be such that $T \leq N$. We know that $(M_{n\wedge T})_{n\geq 0}$ is a martingale, so that $\mathbb{E}[M_{n\wedge T}] = \mathbb{E}[M_0]$ for every $n \geq 0$. It suffices to take $n = N$, since $N \wedge T = T$.

(2) Let us prove

$$|M_{T\wedge n}| \leq |M_0| + \sum_{i=0}^{\infty} |M_{i+1} - M_i| \mathbb{1}_{T>i}. \tag{1}$$

Write

$$M_{T\wedge n} = M_0 + \sum_{i=0}^{n\wedge T-1} (M_{i+1} - M_i) \leq M_0 + \sum_{i=0}^{\infty} (M_{i+1} - M_i) \mathbb{1}_{T>i},$$

and we get (1) by triangular inequality.

Since $\mathbb{E}[T] < \infty$, it follows that $T < \infty$ a.s. As a consequence, $M_{T\wedge n}$ converges almost surely to M_T. In addition, by (1), we are in position to use dominated convergence since $|M_0| + \sum_{i=0}^{\infty} |M_{i+1} -$

$M_i|\mathbb{1}_{T>i}$ is integrable. Indeed using the fact that $\mathbb{1}_{T>i}$ is $\mathcal{F}_i$ measurable, write

$$\begin{aligned}\mathbb{E}[|M_0|]+\sum_{i=0}^{\infty}\mathbb{E}[|M_{i+1}-M_i|\mathbb{1}_{T>i}] &= \mathbb{E}[|M_0|]+\sum_{i=0}^{\infty}\mathbb{E}[\mathbb{E}[|M_{i+1}-M_i|\,|\,\mathcal{F}_i]\,\mathbb{1}_{T>i}] \\ &\le \mathbb{E}[|M_0|]+\sum_{i=0}^{\infty}K\mathbb{E}[\mathbb{1}_{T>i}] \\ &= \mathbb{E}[|M_0|]+K\mathbb{E}[T]<\infty,\end{aligned}$$

where we have used the fact that $\mathbb{E}[Z]=\sum_{i=1}^{\infty}\mathbb{P}(Z\ge i)$ for every non-negative integer valued random variable Z. We thus get $\mathbb{E}[M_{T\wedge n}]\to\mathbb{E}[M_T]$ as $n\to\infty$. Since $\mathbb{E}[M_{T\wedge n}]=\mathbb{E}[M_0]$ for every $n\ge 0$, we get the desired result.

(3) We use (2) with the martingale $M_n=S_n-\mathbb{E}[X_1]n$. We just have to check that there exists $K>0$ such that a.s. we have $\mathbb{E}[|M_{n+1}-M_n||\mathcal{F}_n]\le K$ for every $n\ge 0$. To this end write

$$\mathbb{E}[|M_{n+1}-M_n||\mathcal{F}_n]=\mathbb{E}[|X_{n+1}-\mathbb{E}[X_1]||\mathcal{F}_n]\le 2\mathbb{E}[|X_1|].$$

□

2 Training exercises

Exercise 2. Let $(\mathcal{F}_n)_{n\ge 0}$ be a filtration and let S,T be two stopping times with respect to $(\mathcal{F}_n)_{n\ge 0}$. Let $S,T:\Omega\to\mathbb{N}\cup\{\infty\}$ be $(\mathcal{F}_n)$ stopping times. Prove or disprove with a counter-example the following statements:

(1) $S\vee T$ is a stopping time.

(2) $S\wedge T$ is a stopping time.

(3) $S+T$ is a stopping time.

(4) $S+1$ is a stopping time.

(5) $S-1$ is a stopping time.

Solution:

(1) This is true. Indeed for $n\ge 0$ we have $\{S\vee T\le n\}=\{S\le n\}\cap\{T\le n\}\in\mathcal{F}_n$ for $n\ge 0$ since $\{S\le n\},\{T\le n\}\in\mathcal{F}_n$.

(2) This is true. For $n\ge 0$ we have $\{S\wedge T>n\}=\{S>n\}\cap\{T>n\}$ which is an element of $\mathcal{F}_n$ since $\mathcal{F}_n$ is stable under intersections and $\{S>n\}=\{S\le n\}^c,\{T>n\}=\{T\le n\}^c\in\mathcal{F}_n$. Therefore also

$\{S \wedge T \le n\} = \{S \wedge T > n\}^c \in \mathcal{F}_n$ as required.

(3) This is also true. Indeed, we have

$$\{S+T \le n\} = \bigcup_{k+\ell \le n} \{S \le k\} \cap \{T \le \ell\}.$$

Also $\{S \le k\} \in \mathcal{F}_k \subset \mathcal{F}_n$ and $\{T \le \ell\} \in \mathcal{F}_\ell \subset \mathcal{F}_n$ for $k, \ell \le n$. Thus $\{S+T \le n\} \in \mathcal{F}_n$ for all $n \ge 0$ as required.

(4) This is true. Indeed, for $n \ge 1$ we have $\{S+1 \le n\} = \{S \le n-1\} \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$ and $\{S+1=0\} = \emptyset$.

(5) This is not true. For instance, consider a Bernoulli random variable B with parameter $1/2$ and let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma(B)$ for $n \ge 1$. Then $T := B+1$ is an $(\mathcal{F}_n)$ stopping time but $\{T-1=0\} = \{B=0\} \notin \mathcal{F}_0$.

□

Exercise 3. Let $(\mathcal{F}_n)$ be a filtration and $\mathcal{F}_\infty := \sigma(\cup_n \mathcal{F}_n)$. Let (X_n) be a sequence of integrable random variables such that $X_n \to X$ as $n \to \infty$ both a.s. and in L^1, where X is an integrable random variable. Assume that for all $n \ge 0$, $|X_n| \le Y$ a.s., where Y is a non-negative integrable random variable.

(1) Define $Z_n = \sup_{m \ge n} |X_m - X|$. Show that $Z_n \to 0$ as $n \to \infty$ a.s. and in L^1.

(2) Show that $\mathbb{E}[X_n \mid \mathcal{F}_n] \to \mathbb{E}[X \mid \mathcal{F}_\infty]$ as $n \to \infty$ a.s.

(3) Let (Y_n) and (Z_n) be two independent sequences of independent random variables such that $\mathbb{P}(Y_n = n) = n^{-2} = 1 - \mathbb{P}(Y_n = 0)$ and $\mathbb{P}(Z_n = n) = n^{-1} = 1 - \mathbb{P}(Z_n = 0)$. Set $X_n = Y_n Z_n$ and $\mathcal{A} = \sigma(Z_n : n \ge 0)$.

Show that $X_n \to 0$ as $n \to \infty$ both a.s. and in L^1, but $\mathbb{E}[X_n \mid \mathcal{A}]$ does not converge to 0 almost surely.

Solution:

(1) Since $X_n \to X$ a.s. it is clear that $Z_n \to 0$ a.s. as $n \to \infty$. To see that this convergence is also in L^1 it suffices to prove that (Z_n) is uniformly integrable. This follows from the fact that $Z_n \le 2Y$ for all $n \ge 1$ a.s. with Y is integrable.

(2) For $m \le n$ a.s.

$$\begin{aligned} |\mathbb{E}[X_n \mid \mathcal{F}_n] - \mathbb{E}[X \mid \mathcal{F}_\infty]| &\le \mathbb{E}[|X - X_n| \mid \mathcal{F}_n] + |\mathbb{E}[X \mid \mathcal{F}_n] - \mathbb{E}[X \mid \mathcal{F}_\infty]| \\ &\le \mathbb{E}[Z_m \mid \mathcal{F}_n] + |\mathbb{E}[X \mid \mathcal{F}_n] - \mathbb{E}[X \mid \mathcal{F}_\infty]|. \end{aligned}$$

As seen in the lecture, the right hand side converges to $\mathbb{E}[Z_m \mid \mathcal{F}_\infty]$ a.s. and in L^1 as $n \to \infty$ for fixed m. Therefore

$$\limsup_{n\to\infty} |\mathbb{E}[X_n \mid \mathcal{F}_n] - \mathbb{E}[X \mid \mathcal{F}_\infty]| \le \inf_{m\ge 1} \mathbb{E}[Z_m \mid \mathcal{F}_\infty] \quad \text{a.s.}$$

The claim follows since $\mathbb{E}[Z_m \mid \mathcal{F}_\infty] \to 0$ as $m \to \infty$ a.s. by dominated convergence for conditional expectations.

(3) We have

$$\mathbb{E}(X_n) = \frac{1}{n} \quad \text{and} \quad \mathbb{E}[X_n \mid \mathcal{A}] = \mathbb{E}[Y_n] Z_n = \frac{1}{n} Z_n \quad \text{a.s.}$$

As a consequence $X_n \to 0$ in L^1. Also, since $\mathbb{P}(X_n > 0) \le \mathbb{P}(Y_n, Z_n > 0) = \frac{1}{n^2}$ and since $\sum_{n\ge 1} \frac{1}{n^2} < \infty$, it follows that a.s. $X_n = 0$ for n sufficiently large, so that $X_n \to 0$ a.s.

Finally

$$\sum_{n\ge 1} \mathbb{P}(Z_n n^{-1} \ge 1) \ge \sum_{n\ge 1} \mathbb{P}(Z_n = n) = \sum_{n \ge 1} \frac{1}{n} = \infty$$

and so by Borel-Cantelli we have $\mathbb{E}[X_n \mid \mathcal{A}] \not\to 0$ almost surely.

□

Exercise 4. Let T be a stopping time for a filtration $(\mathcal{F}_n)_{n\ge 0}$. Assume that there exit $\varepsilon > 0$ and $n_0 \ge 1$ such that for every $n \ge 0$, almost surely

$$\mathbb{P}(T \le n + n_0 | \mathcal{F}_n) > \varepsilon.$$

(1) Show that for every $k \ge 0$ we have $\mathbb{P}(T \ge k n_0) \le (1-\varepsilon)^k$.

(2) Show that T is almost surely finite and that $\mathbb{E}[T] < \infty$.

Solution:

(1) We argue by induction on k. For $k = 0$, the result is true. Assume that $\mathbb{P}(T \ge k n_0) \le (1-\varepsilon)^k$. Then write, using the fact that $\mathbb{1}_{T \ge k n_0}$ is measurable with respect to $\mathcal{F}_{k n_0}$,

$$\mathbb{P}(T \ge (k+1) n_0) = \mathbb{E}\left[\mathbb{1}_{T\ge k n_0, T \ge (k+1) n_0}\right] = \mathbb{E}\left[\mathbb{1}_{T \ge k n_0} \mathbb{P}\left(T \ge k n_0 + n_0 | \mathcal{F}_{k n_0}\right)\right] \le \mathbb{E}\left[\mathbb{1}_{T \ge k n_0} (1-\varepsilon)\right]$$

which is at most $(1-\varepsilon)^{k+1}$ by induction hypothesis.

(2) Write

$$\mathbb{E}[T] = \sum_{k=0}^{\infty} \mathbb{E}\left[T \mathbb{1}_{k n_0 \le T < (k+1) n_0}\right] \le \sum_{k=0}^{\infty} (k+1) n_0 \mathbb{P}(k n_0 \le T) \le \sum_{k=0}^{\infty} (k+1) n_0 (1-\varepsilon)^k < \infty.$$

A fortiori, this shows that $T < \infty$ a.s.

□

Exercise 5. Let $(M_n)_{n\geq 0}$ be a uniformly integrable martingale with respect to a filtration $(\mathcal{F}_n)_{n\geq 0}$.

(1) Is it true that the collection $\{M_T : T \text{ stopping time with respect to } (\mathcal{F}_n)_{n\geq 0}\}$ is uniformly integrable?

(2) Let T be a stopping time. Is it true that $(M_{n\wedge T})_{n\geq 0}$ is a uniformly integrable martingale? Justify your answer.

Solution:

(1) Yes it is true. It follows from the following two facts seen in the lecture : $M_T = \mathbb{E}[Z|\mathcal{F}_T]$ and for any collection of σ-fields $(\mathcal{A}_i)_{i\in I}$ the collection $(\mathbb{E}[Z|\mathcal{A}_i])_{i\in I}$ is uniformly integrable.

(2) Yes it is true, as a consequence of (1) since $n \wedge T$ is a stopping time for every $n \geq 0$.

□

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 6. Let T be a stopping time with respect to a filtration $(\mathcal{F}_n)_{n\geq 0}$ with $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}$. Recall that the σ-field $\mathcal{F}_T$ is defined by

$$\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T = n\} \in \mathcal{F}_n \ \forall n \geq 0\}.$$

(1) Let S be a stopping time with respect to the filtration $(\mathcal{F}_n)_{n\geq 0}$ such that $S \leq T$. Show that $\mathcal{F}_S \subset \mathcal{F}_T$.

(2) Show that T is $\mathcal{F}_T$ measurable.

(3) Here we assume that $(X_n)_{n\geq 0}$ is a sequence of random variables and that $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$. Show that $\mathcal{F}_T = \sigma(X_{T\wedge n} : n \geq 0)$.

Solution:

(1) Take $A \in \mathcal{F}_S$ and $n \geq 0$. We show that $A \cap \{T = n\} \in \mathcal{F}_n$. To this end, using $S \leq T$ write

$$A \cap \{T = n\} = \left(\bigcup_{s=0}^{n} A \cap \{T = s\} \right) \cap \{T = n\} \in \mathcal{F}_n.$$

(2) It is enough to show that for every $k \geq 0$, $\{T = k\} \in \mathcal{F}_T$. To this end, take $n \geq 0$ and write $\{T = k\} \cap \{T = n\} = \varnothing \in \mathcal{F}_n$ for $n \neq k$ and $\{T = k\} \cap \{T = k\} \in \mathcal{F}_n$ for $n = k$ by definition of a stopping time.

(3) We argue by double inclusion.

∗ First, we have seen in the lecture that $X_{n\wedge T}$ is $\mathcal{F}_{n\wedge T}$ measurable. Since $T \wedge n \leq T$, we have $\mathcal{F}_{n\wedge T} \subset \mathcal{F}_T$ by (1), so $X_{n\wedge T}$ is $\mathcal{F}_T$ measurable. Thus

$$\sigma(X_{n\wedge T} : n \geq 0) \subset \mathcal{F}_T.$$

∗ We now show that

$$\mathcal{F}_T \subset \sigma(X_{n\wedge T} : n \geq 0).$$

We show by strong induction that

$$\forall k \geq 0, \forall A \in \mathcal{F}_T, \quad A \cap \{T = k\} \in \sigma(X_{n\wedge T} : n \geq 0).$$

– $k = 0$. For $A \in \mathcal{F}_T$,

$$A \cap \{T = 0\} \in \mathcal{F}_0 = \sigma(X_0) = \sigma(X_{0\wedge T}) \subset \sigma(X_{n\wedge T} : n \geq 0)$$

– Assume that for every $0 \leq i \leq k-1$ and for every $A \in \mathcal{F}_T$ we have $A \cap \{T = i\} \in \sigma(X_{n\wedge T} : n \geq 0)$. Take $A \in \mathcal{F}_T$. We have

$$A \cap \{T = k\} \in \mathcal{F}_k = \sigma(X_0, \ldots, X_k),$$

so by the Doob-Dynkin lemma applied with the $\mathbb{R}^{k+1}$-valued random variable $(X_0, \ldots, X_k)$, there exists a measurable function $f : \mathbb{R}^{k+1} \to \mathbb{R}$ such that

$$\mathbb{1}_{A\cap\{T=k\}} = f(X_0, X_1, \ldots, X_k).$$

As $\mathbb{1}_{\{T=k\}} = \mathbb{1}_{\{T=k\}}\mathbb{1}_{\{T\geq k\}}$ we get

$$\mathbb{1}_{A\cap\{T=k\}} = f(X_1, \ldots, X_k)\mathbb{1}_{\{T\geq k\}} = f(X_{0\wedge T}, X_{1\wedge T}, \ldots, X_{k\wedge T})\mathbb{1}_{\{T\geq k\}}.$$

Because of the induction hypothesis (applied with $A = \Omega$), we have

$$\{T \geq k\} = \{T \leq k-1\}^c \in \sigma(X_{n\wedge T} : n \geq 0),$$

and therefore it follows that $\mathbb{1}_{A\cap\{T=k\}}$ is $\sigma(X_{n\wedge T} : n \geq 0)$-measurable; this is equivalent to $A \cap \{T = k\} \in \sigma(X_{n\wedge T} : n \geq 0)$.

□

Exercise 7. Let (M_n) be a martingale with respect to a filtration $(\mathcal{F}_n)$ and let S and T be stopping times. Show that for every $n \geq 0$ we almost surely have

$$\mathbb{E}(M_{n\wedge S} \mid \mathcal{F}_T) = M_{n\wedge S\wedge T}$$

Solution:

First of all, for $n \geq 0$, $M_{S\wedge T\wedge n}$ is measurable with respect to $\mathcal{F}_{S\wedge T\wedge n}$, so is is measurable with respect to $\mathcal{F}_T$ (Exercise 6 (1))

We now fix $A \in \mathcal{F}_T$ and show that

$$\mathbb{E}[M_{n\wedge S}\mathbb{1}_A] = \mathbb{E}[M_{n\wedge S\wedge T}\mathbb{1}_A]$$

which will imply the result.

To this end start with wriing

$$\mathbb{E}[M_{n\wedge S}\mathbb{1}_A] = \sum_{k=0}^{n} \mathbb{E}\left[M_{n\wedge S}\mathbb{1}_{A\cap\{T=k\}}\right] + \mathbb{E}\left[M_{n\wedge S}\mathbb{1}_{A\cap\{T>n\}}\right]$$

We know that $(M_{n\wedge S})_{n\geq 0}$ is a martingale and since $A\cap\{T=k\} \in \mathcal{F}_k$ (by definition) we have for $k \leq n$,

$$\mathbb{E}\left[M_{n\wedge S}\mathbb{1}_{A\cap\{T=k\}}\right] = \mathbb{E}\left[\mathbb{E}[M_{n\wedge S} \mid \mathcal{F}_k]\mathbb{1}_{A\cap\{T=k\}}\right] = \mathbb{E}\left[M_{k\wedge S}\mathbb{1}_{A\cap\{T=k\}}\right].$$

Therefore

$$\mathbb{E}[M_{n\wedge S}\mathbb{1}_A] = \sum_{k=0}^{n} \mathbb{E}\left[M_{k\wedge S}\mathbb{1}_{A\cap\{T=k\}}\right] + \mathbb{E}\left[M_{n\wedge S}\mathbb{1}_{A\cap\{T>n\}}\right] = \mathbb{E}[M_{n\wedge S\wedge T}\mathbb{1}_A]$$

which completes the argument. □

Exercise 8. Let $f : [0,1] \to \mathbb{R}$ be a Lipschitz function, i.e. there exists $K > 0$ such that $|f(x)-f(y)| \leq K|x-y|$ for all $x, y \in [0,1]$. Let $f_n' : [0,1] \to \mathbb{R}$ be defined by

$$f_n'(x) = \begin{cases} 2^n\left(f\left(\frac{i+1}{2^n}\right) - f\left(\frac{i}{2^n}\right)\right) & : x \in [i/2^n, (i+1)/2^n)\,, \\ 0 & : x = 1\,. \end{cases}$$

Note that f_n' is the derivative of the piecewise linear extension of $f|_{(2^{-n}\mathbb{Z})\cap[0,1]}$ to $[0,1]$.

(1) Show that $f_n' \to f'$ almost everywhere and in L^1 (with respect to the Lebesgue measure) as $n \to \infty$ for some integrable function $f' : [0,1] \to \mathbb{R}$.

Hint. Use the martingale convergence theorem after defining a suitable probability space together with a martingale on it.

(2) Deduce that

$$f(x) - f(0) = \int_0^x f'(y)\,dy \quad \text{for all } x \in [0,1]\,.$$

Solution:

(1) Let $\Omega = [0,1]$, $\mathcal{F} = \mathcal{B}([0,1))$ and let $\mathbb{P}$ be the Lebesgue measure on $[0,1]$. We also define the filtration

$$\mathcal{F}_n = \sigma([i2^{-n},(i+1)2^{-n})\colon i = 0,\dots,2^n - 1)\,.$$

Clearly the sequence (f_n') is adapted with respect to $(\mathcal{F}_n)$. Moreover, $|f_n'| \le K$ by the Lipschitz property, so (f_n') is bounded in L^∞ and in particular uniformly integrable. Lastly, let us verify the martingale property for (f_n'). We have

$$\mathbb{E}[f_{n+1}' \mid \mathcal{F}_n] = \sum_{i=0}^{2^{n+1}-1} 2^{n+1}\left(f\left(\frac{i+1}{2^{n+1}}\right) - f\left(\frac{i}{2^{n+1}}\right)\right)\mathbb{P}([i/2^{n+1},(i+1)/2^{n+1}) \mid \mathcal{F}_n)$$

almost surely. They key is now that if $a = j/2^n$ and $b = (j+1)/2^n$ then

$$\mathbb{P}([a, a+(b-a)/2) \mid \mathcal{F}_n) = \mathbb{P}([a+(b-a)/2, b) \mid \mathcal{F}_n) = \frac{1}{2}1_{[a,b)} \quad \text{a.s.}$$

This follows since $\mathcal{F}_n$ is a σ-algebra which is generated by a collection of disjoint sets and hence the conditional expectation is explicit. Using this observation, the terms in the expression $\mathbb{E}[f_{n+1}' \mid \mathcal{F}_n]$ with $i = 2j$ and with $i = 2j+1$ combine and we see that

$$\mathbb{E}[f_{n+1}' \mid \mathcal{F}_n] = f_n' \quad \text{a.s.}$$

Hence (f_n') is a uniformly integrable martingale and hence converges to an integrable limit f' almost surely and in L^1.

(2) For $n \ge m \ge 0$ and $i \in \{0,\dots,2^m - 1\}$ we have by definition that

$$f(i/2^m) - f(0) = \int_0^{i/2^m} f_n'(y)\,dy \to \int_0^{i/2^m} f'(y)\,dy \quad \text{as } n \to \infty\,.$$

Therefore the result holds for all x of the form $x = i/2^m$ and the full result follows by continuity of f and integrability of f' (using dominated convergence).

□

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 9. You throw a fair six-sided die until you get 6. What is the expected number of throws (including the throw giving 6) conditioned on the event that all throws gave even numbers?

Solution:

See https://www.yichijin.com//files/elchanan.pdf □