Week 11: L^p martingales, $p > 1$.

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 4/12/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

***

1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d random variables L^2 with $\mathbb{E}[X_1] = 0$, and set $\sigma^2 = \mathsf{Var}(X_1)$. Set $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Set also $M_n = S_n^2 - n\sigma^2$ for $n \geq 0$ and $\mathcal{F}_n = \sigma(M_0, \ldots, M_n)$. Let T be a $(\mathcal{F}_n)$ stopping time with $\mathbb{E}[T] < \infty$.

(1) Show that (M_n) is a $(\mathcal{F}_n)$ martingale.

(2) Show that $\mathbb{E}\left[S_{T\wedge n}^2\right] = \sigma^2 \mathbb{E}[T \wedge n]$ for every $n \geq 0$.

(3) Show that $(S_{T\wedge n})_{n\geq 0}$ is bounded in L^2.

(4) Conclude that $\mathbb{E}[S_T^2] = \sigma^2 \mathbb{E}[T]$.

2 Training exercises

Exercise 2. Let $(M_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$ martingale bounded in L^p with $p > 1$. Show that

$$\mathbb{E}\left[\sup_{n\geq 0} |M_n|^p\right] \leq \left(\frac{p}{p-1}\right)^p \sup_{n\geq 0} \mathbb{E}[|M_n|^p].$$

Exercise 3. Let $(X_i)_{i\geq 1}$ be i.i.d. random variables with values in $\{-1, 1\}$ where we write $\mathbb{P}(X_i = 1) = p$ and assume that $p \in (0, 1/2)$. Moreover, define $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. For $n \geq 0$ we set

$$M_n = \left(\frac{1}{p} - 1\right)^{S_n}.$$

For $n \geq 1$ set $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Recall from Exercise Sheet 9 Exercise 3 that (M_n) is a $(\mathcal{F}_n)_{n\geq 0}$ martingale.

(1) Show that for every $a > 0$ we have

$$\mathbb{P}\left(\sup_{n\geq 0} M_n \geq a\right) \leq \frac{1}{a}.$$

(2) Show that for every $k \geq 0$ we have

$$\mathbb{P}\left(\sup_{n\geq 0} S_n \geq k\right) \leq \left(\frac{p}{1-p}\right)^k$$

(3) Deduce that $\mathbb{E}\left[\sup_{n\geq 0} S_n\right] \leq \frac{p}{1-2p}$.

Exercise 4. (Azuma's inequality) Let M_n be a martingale starting from 0 with respect to a filtration $(\mathcal{F}_n)$ with $|M_n - M_{n-1}| \leq c_n$ for all $n \geq 1$ and finite deterministic constants $c_n < \infty$.

(1) Show that if Y is a random variable with mean 0 and $|Y| \leq c$ then for $\theta \in \mathbb{R}$,

$$\mathbb{E}(e^{\theta Y}) \leq \cosh(\theta c) \leq e^{\theta^2 c^2/2}.$$

Hint. Use the convexity of $y \mapsto e^{\theta y}$ on $[-c, c]$.

(2) Show that for $\theta \in \mathbb{R}$,

$$\mathbb{E}(e^{\theta M_n}) \leq e^{\theta^2 \sigma_n^2/2}$$

where $\sigma_n^2 = c_1^2 + \cdots + c_n^2$.

(3) Deduce that for $x \geq 0$,

$$\mathbb{P}\left(\sup_{0\leq k\leq n} M_k \geq x\right) \leq e^{-x^2/(2\sigma_n^2)}.$$

Hint. Introduce $N_n = \exp(\theta M_n - \theta^2 \sigma_n^2/2)$.

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 5. Let $(X_n)_{n\geq 1}$ be a sequence of independent non-negative random variables with $\mathbb{E}[X_n] = 1$ for every $n \geq 1$ (the random variables do not necessarily have the same law). Set $M_0 = 1$ and for $n \geq 1$:

$$M_n = \prod_{k=1}^{n} X_k.$$

(1) Show that $(M_n)_{n\geq 1}$ is a martingale which converges a.s. to a random variable denoted by M_∞.

For $k \geq 1$ set $a_k = \mathbb{E}\left[\sqrt{X_k}\right]$ which belongs to $(0,1]$ (by the Cauchy-Schwarz inequality). Define $N_0 = 1$ and for $n \geq 1$

$$N_n = \prod_{k=1}^{n} \frac{\sqrt{X_k}}{a_k}.$$

(2) Using the process (N_n), show that the following five conditions are equivalent:

   (a) $\mathbb{E}[M_\infty] = 1$;
   (b) $M_n \to M_\infty$ in L^1;
   (c) the martingale (M_n) is uniformly integrable;
   (d) $\prod_{k=1}^{\infty} a_k > 0$;
   (e) $\sum_{k=1}^{\infty} (1 - a_k) < \infty$.

   Also show that if one of these conditions are not satisfied, then $M_\infty = 0$ a.s.

(3) Is it true that a supermartingale bounded in L^p converges in L^p? Justify your answer.

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 6. Suppose your friend is turning over cards from a face-down shuffled deck, and at any point you can call ”Next”, and if the next card is red, you win a prize.

Clearly, if you immediately shout “Next”, your chances of winning are 1/2. Can you devise a strategy that does better than 1/2 – for example, waiting until there are slightly more red cards remaining and then calling ”Next”, even though you might never reach a state where there are slightly more red cards?