Week 11: L^p martingales, $p > 1$.

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 4/12/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d random variables L^2 with $\mathbb{E}[X_1] = 0$, and set $\sigma^2 = \mathrm{Var}(X_1)$. Set $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Set also $M_n = S_n^2 - n\sigma^2$ for $n \geq 0$ and $\mathcal{F}_n = \sigma(M_0, \ldots, M_n)$. Let T be a $(\mathcal{F}_n)$ stopping time with $\mathbb{E}[T] < \infty$.

(1) Show that (M_n) is a $(\mathcal{F}_n)$ martingale.

(2) Show that $\mathbb{E}\left[S^2_{T\wedge n}\right] = \sigma^2 \mathbb{E}[T \wedge n]$ for every $n \geq 0$.

(3) Show that $(S_{T\wedge n})_{n\geq 0}$ is bounded in L^2.

(4) Conclude that $\mathbb{E}[S_T^2] = \sigma^2 \mathbb{E}[T]$.

Solution:

(1) M_n is integrable since S_n^2 is integrable (beause X_n is in L^2). M_n is $\mathcal{F}_n$ measurable by definition of $\mathcal{F}_n$. Then write

$$\begin{aligned}\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] &= \mathbb{E}[(S_n + X_{n+1})^2 \mid \mathcal{F}_n] - (n+1)\sigma^2 \\ &= \mathbb{E}[S_n^2 \mid \mathcal{F}_n] + \mathbb{E}[S_n X_{n+1} \mid \mathcal{F}_n] + \mathbb{E}[X_{n+1}^2 \mid \mathcal{F}_n] - (n+1)\sigma^2\end{aligned}$$

Since X_{n+1} is independent of $\mathcal{F}_n$ we have $\mathbb{E}[X_{n+1}^2 \mid \mathcal{F}_n] = \mathbb{E}[X_{n+1}^2] = \sigma^2$. By the tower property,

$$\mathbb{E}[S_n X_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[\mathbb{E}[S_n X_{n+1} \mid \mathcal{F}_n, S_n] \mid \mathcal{F}_n] = \mathbb{E}[S_n \mathbb{E}[X_{n+1} \mid \mathcal{F}_n, S_n] \mid \mathcal{F}_n]$$

and $\mathbb{E}[X_{n+1} \mid \mathcal{F}_n, S_n] = \mathbb{E}[X_{n+1}] = 0$ since X_{n+1} is independent from $\sigma(\mathcal{F}_n, S_n)$. We conclude that

$$\begin{aligned}\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] &= S_n^2 + 0 + \sigma^2 - (n+1)\sigma^2 \\ &= M_n.\end{aligned}$$

For the third equality we have used the fact that S_n is $\mathcal{F}_n$ measurable and that X_{n+1} is independent of $\mathcal{F}_n$.

(2) Since T is a stopping time, $(M_{T\wedge n})_{n\geq 0}$ is a $(\mathcal{F}_n)$-martingale, so $\mathbb{E}[M_{T\wedge n}] = \mathbb{E}[M_0] = 0$ for every $n \geq 0$, which gives the result.

(3) By (2), $\mathbb{E}\left[S_{T\wedge n}^2\right] \leq \sigma^2 \mathbb{E}[T]$, which gives the result.

(4) By monotone convergence, $\mathbb{E}[T \wedge n] \to \mathbb{E}[T]$ as $n \to \infty$. To show that $\mathbb{E}\left[S_{T\wedge n}^2\right] \to \mathbb{E}[S_T^2]$ as $n \to \infty$, we use the fact that (S_n), and thus also $(S_{T\wedge n})$, is a $(\mathcal{F}_n)$ martingale. Since $\mathbb{E}[T] < \infty$, we have $T < \infty$ almost surely, so $S_{T\wedge n}$ converges almost surely to S_T. Also, since $(S_{T\wedge n})_{n\geq 0}$ is bounded in L^2, the previous convergence holds in L^2, which implies that $\mathbb{E}\left[S_{T\wedge n}^2\right] \to \mathbb{E}[S_T^2]$ and gives the desired result.

□

2 Training exercises

Exercise 2. Let $(M_n)_{n\geq 0}$ be a $(\mathcal{F}_n)_{n\geq 0}$ martingale bounded in L^p with $p > 1$. Show that

$$\mathbb{E}\left[\sup_{n\geq 0} |M_n|^p\right] \leq \left(\frac{p}{p-1}\right)^p \sup_{n\geq 0} \mathbb{E}[|M_n|^p].$$

Solution:

We have seen in the lecture that

$$\mathbb{E}\left[\left(\sup_{n\geq 0} |M_n|\right)^p\right] \leq \left(\frac{p}{p-1}\right)^p \sup_{n\geq 0} \mathbb{E}[|M_n|^p].$$

Since

$$\left(\sup_{n\geq 0} |M_n|\right)^p = \sup_{n\geq 0} |M_n|^p$$

the desired result follows. □

Exercise 3. Let $(X_i)_{i\geq 1}$ be i.i.d. random variables with values in $\{-1, 1\}$ where we write $\mathbb{P}(X_i = 1) = p$ and assume that $p \in (0, 1/2)$. Moreover, define $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. For $n \geq 0$ we set

$$M_n = \left(\frac{1}{p} - 1\right)^{S_n}.$$

For $n \geq 1$ set $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

Recall from Exercise Sheet 9 Exercise 3 that (M_n) is a $(\mathcal{F}_n)_{n\geq 0}$ martingale.

(1) Show that for every $a > 0$ we have

$$\mathbb{P}\left(\sup_{n\geq 0} M_n \geq a\right) \leq \frac{1}{a}.$$

(2) Show that for every $k \geq 0$ we have

$$\mathbb{P}\left(\sup_{n\geq 0} S_n \geq k\right) \leq \left(\frac{p}{1-p}\right)^k$$

(3) Deduce that $\mathbb{E}\left[\sup_{n\geq 0} S_n\right] \leq \frac{p}{1-2p}$.

Solution:

Note that $|M_n| = M_n$ since $M_n \geq 0$.

(1) By Doob's maximal inequality, for $\ell \geq n$ we have

$$a\mathbb{P}\left(\sup_{0\leq n\leq \ell} M_n \geq a\right) \leq \mathbb{E}[M_\ell] = 1.$$

But $\sup_{0\leq n\leq \ell} M_n \to \sup_{n\geq 0} M_n$ in an increasing way when $\ell \to \infty$, so $\mathbb{P}\left(\sup_{0\leq n\leq \ell} M_n \geq a\right) \to \mathbb{P}\left(\sup_{n\geq 0} M_n \geq a\right)$ as $\ell \to \infty$, and we get the desired result.

(2) Observe that since $r \mapsto ((1-p)/p)^r$ is increasing, we have

$$\left\{\sup_{n\geq 0} S_n \geq k\right\} = \left\{\sup_{n\geq 0} M_n \geq \left(\frac{1-p}{p}\right)^k\right\}$$

and the result follows from (1).

(3) We use the previously seen fact that if $Z \geq 0$ is an integer valued random variable we have $\mathbb{E}[Z] = \sum_{k=1}^{\infty} \mathbb{P}(Z \geq k)$, so that by (2)

$$\mathbb{E}\left[\sup_{n\geq 0} S_n\right] \leq \sum_{k=1}^{\infty} \left(\frac{p}{1-p}\right)^k = \frac{p}{1-p} \cdot \frac{1}{1-\frac{p}{1-p}} = \frac{p}{1-2p}.$$

□

Exercise 4. (Azuma's inequality) Let M_n be a martingale starting from 0 with respect to a filtration $(\mathcal{F}_n)$ with $|M_n - M_{n-1}| \leq c_n$ for all $n \geq 1$ and finite deterministic constants $c_n < \infty$.

(1) Show that if Y is a random variable with mean 0 and $|Y| \leq c$ then for $\theta \in \mathbb{R}$,

$$\mathbb{E}(e^{\theta Y}) \leq \cosh(\theta c) \leq e^{\theta^2 c^2/2}.$$

Hint. Use the convexity of $y \mapsto e^{\theta y}$ on $[-c, c]$.

(2) Show that for $\theta \in \mathbb{R}$,

$$\mathbb{E}(e^{\theta M_n}) \leq e^{\theta^2 \sigma_n^2/2}$$

where $\sigma_n^2 = c_1^2 + \cdots + c_n^2$.

(3) Deduce that for $x \geq 0$,

$$\mathbb{P}\left(\sup_{0\leq k\leq n} M_k \geq x\right) \leq e^{-x^2/(2\sigma_n^2)}.$$

Hint. Introduce $N_n = \exp(\theta M_n - \theta^2\sigma_n^2/2)$.

Solution:

(1) Using the hint, we obtain

$$e^{\theta Y} \leq \frac{Y+c}{2c}e^{\theta c} + \frac{-Y+c}{2c}e^{-\theta c}.$$

Taking expectations and using that $\mathbb{E}(Y) = 0$ yields $\mathbb{E}(e^{\theta Y}) \leq \cosh(\theta c)$. Finally for $x \in \mathbb{R}$,

$$\cosh(x) = \sum_{k\geq 0}\frac{x^{2k}}{(2k)!} \leq \sum_{k\geq 0}\frac{x^{2k}}{2^k k!} = e^{x^2/2}$$

and the second inequality follows by taking $x = \theta c$.

(2) We prove the claim using induction. The $n = 0$ claim is obvious. For the induction step, we observe that

$$\mathbb{E}\left[e^{\theta M_{n+1}}\right] = \mathbb{E}\left[e^{\theta M_n}\mathbb{E}\left[e^{\theta(M_{n+1}-M_n)} \mid \mathcal{F}_n\right]\right] \quad \text{a.s.} \tag{1}$$

Since $\mathbb{E}[M_{n+1} - M_n \mid \mathcal{F}_n] = 0$ and $|M_{n+1} - M_n| \leq c_{n+1}$ the same argument as in (1), now in the setting of conditional expectations yields that

$$\mathbb{E}\left[e^{\theta(M_{n+1}-M_n)} \mid \mathcal{F}_n\right] \leq e^{\theta^2 c_{n+1}^2/2} \quad \text{a.s.}.$$

Substituting this into (**??**) and using the induction hypothesis yields the induction step.

(3) Let $N_n = \exp(\theta M_n - \theta^2\sigma_n^2/2)$. We claim that (N_n) is a supermartingale with respect to the filtration $(\mathcal{F}_n)$. It is clearly $\mathcal{F}_n$-measurable and integrability is shown in (2). For the supermartingale property observe that for $n \geq 0$,

$$\mathbb{E}[N_{n+1} \mid \mathcal{F}_n] = e^{\theta M_n - \theta^2\sigma_{n+1}^2/2}\mathbb{E}\left[e^{\theta(M_{n+1}-M_n)} \mid \mathcal{F}_n\right] \leq e^{\theta M_n - \theta^2\sigma_{n+1}^2/2}e^{\theta^2 c_{n+1}^2/2} = N_n \quad \text{a.s.}$$

as we saw as part of the proof of (2). Now let us define the following stopping time $T = \inf\{n \geq 0 : M_n \geq x\}$.

Then, by the same method of proof as for martingales, the stopped process $(N_{n\wedge T})_{n\geq 0}$ is a supermartingale, so for every $n \geq 0$ we get

$$\mathbb{E}[N_{n\wedge T}] \leq \mathbb{E}[N_0] = 1 .$$

Moreover $N_{n\wedge T} \geq N_T \mathbb{1}_{T\leq n} \geq e^{\theta x - \theta^2\sigma_n^2/2}\mathbb{1}_{T\leq n}$ for $\theta \geq 0$. Combing everything yields

$$e^{\theta x-\theta^2\sigma_n^2/2}\mathbb{P}(T \leq n) \leq 1 .$$

We now take $\theta = x/\sigma_n^2$ (which makes the inequality strongest) and obtain the claim; indeed $\mathbb{P}(T \leq n)$ is exactly $\mathbb{P}\left(\sup_{0\leq k\leq n} M_k \geq x\right)$.

□

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 5. Let $(X_n)_{n\geq 1}$ be a sequence of independent non-negative random variables with $\mathbb{E}[X_n] = 1$ for every $n \geq 1$ (the random variables do not necessarily have the same law). Set $M_0 = 1$ and for $n \geq 1$:

$$M_n = \prod_{k=1}^{n} X_k.$$

(1) Show that $(M_n)_{n\geq 1}$ is a martingale which converges a.s. to a random variable denoted by M_∞.

For $k \geq 1$ set $a_k = \mathbb{E}\left[\sqrt{X_k}\right]$ which belongs to $(0, 1]$ (by the Cauchy-Schwarz inequality). Define $N_0 = 1$ and for $n \geq 1$

$$N_n = \prod_{k=1}^{n} \frac{\sqrt{X_k}}{a_k}.$$

(2) Using the process (N_n), show that the following five conditions are equivalent:

(a) $\mathbb{E}[M_\infty] = 1$;

(b) $M_n \to M_\infty$ in L^1;

(c) the martingale (M_n) is uniformly integrable;

(d) $\prod_{k=1}^{\infty} a_k > 0$;

(e) $\sum_{k=1}^{\infty}(1 - a_k) < \infty$.

Also show that if one of these conditions are not satisfied, then $M_\infty = 0$ a.s.

(3) Is it true that a supermartingale bounded in L^p converges in L^p? Justify your answer.

Solution:

(1) Set $\mathcal{F}_n = \sigma(M_0, \dots, M_n)$. Observe that $M_n \geq 0$ and by independence

$$\mathbb{E}[M_n] = \prod_{k=1}^{n} \mathbb{E}[X_k] = 1 < \infty.$$

In addition,

$$\mathbb{E}[M_{n+1} \mid \mathcal{F}_n] = \mathbb{E}[M_n X_{n+1} \mid \mathcal{F}_n] = M_n \mathbb{E}[X_{n+1}] = M_n.$$

(2)
- We have seen the equivalence $(b) \iff (c)$ in the lecture.
- The fact that $(d) \iff (e)$ is a result from real analysis, which comes from the fact that $\ln(1-x) \sim -x$ as $x \to 0$ and if $a_n \sim b_n$ with all the (a_n) having the same sign, then $\sum_n a_n$ is convergent if and only if $\sum_n b_n$ is convergent.
- for $(b) \implies (a)$, this comes from the fact that convergence in L^1 implies convergence of expectations.
- The fact that $(a) \implies (b)$ comes from Scheffé's Lemma (Exercise sheet 7, exercise 2).
- for $(d) \implies (a)$, we note that (N_n) is a non-negative martingale, which converges a.s. to a random variable denoted by N_∞. If $\prod_{k=1}^{\infty} a_k > 0$, this implies that

$$\mathbb{E}[N_n^2] = \frac{\mathbb{E}[M_n]}{\prod_{k=1}^{n} a_k} \leq \frac{1}{\prod_{k=1}^{\infty} a_k} < \infty.$$

  As a consequence, (N_n) is bounded in L^2 and converges in L^2 to $\sqrt{M_\infty}/\prod_{k=1}^{\infty} a_k$. As a consequence, $\mathbb{E}[N_n^2] \to \mathbb{E}[M_\infty^2]/(\prod_{k=1}^{\infty} a_k)^2$, so $1 = \mathbb{E}[M_n] \to \mathbb{E}[M_\infty]$.
- for $(a) \implies (d)$, we argue by contraposition. Assume that $\prod_{k=1}^{\infty} a_k = 0$. Then as above the non-negative martingale (N_n) converges a.s. to a random variable denoted by N_∞. Since $N_\infty < \infty$ and $\prod_{k=1}^{n} a_k \to 0$, we must have $M_n \to 0$ a.s. so that $M_\infty = 0$ a.s.

(3) We use the previous questions to give a counterexample in the case $p = 2$. Set

$$\mathbb{P}(X_n) = \begin{cases} \frac{(n+1)^2}{n^2} & \text{with probability } \frac{n^2}{(n+1)^2} \\ 0 & \text{with probability } 1 - \frac{n^2}{(n+1)^2}, \end{cases}$$

so that $\mathbb{E}[X_n] = 1$ and $\mathbb{E}\left[\sqrt{X_n}\right] = \frac{n}{n+1}$. Finally, set $S_n = \sqrt{M_n}$.

The computations of question (2) show that (S_n) is a non-negative supermartingale bounded in L^2. But since $\sum_{n \geq 1}(1 - \frac{n}{n+1}) = \infty$, $M_n \to 0$ almost surely, so $\mathbb{E}[S_n^2] \to 0$, so S_n does not converge in L^2.

Remark. By taking $(-S_n)$ we get a (non-positive) submartingale bounded in L^2 which does not converge in L^2.

□

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 6. Suppose your friend is turning over cards from a face-down shuffled deck, and at any point you can call ”Next”, and if the next card is red, you win a prize.

Clearly, if you immediately shout “Next”, your chances of winning are 1/2. Can you devise a strategy that does better than 1/2 – for example, waiting until there are slightly more red cards remaining and then calling ”Next”, even though you might never reach a state where there are slightly more red cards?

Solution:

The answer is no: every strategy has probability 1/2 of winning.

To see this, let R_n denote the number of red cards remaining the deck after n cards have been shown. Set $\mathcal{F}_n = \sigma(R_0, \ldots, R_n)$ and $M_n = \frac{R_n}{52-n}$ the fraction of remaining red cards. We claim that (M_n) is a martingale. Indeed, given $\mathcal{F}_n$, the probability that the next card is red is M_n, so

$$\mathbb{E}[R_{n+1} \mid \mathcal{F}_n] = R_n - \frac{R_n}{52-n} = \frac{51-n}{52-n} R_n,$$

so that $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n$. Since $M_0 = 1/2$, this martingale has mean 1/2.

Now consider any strategy and let N be number of cards after which “Next” has been called. Since $(M_{n \wedge N})_{n \geq 0}$ is a bounded martingale and thus uniformly integrable, the optional stopping theorem implies $\mathbb{E}[M_N] = 1/2$. Denote by W the event of winning. It's probability is the probability that the next card is red, which given $\mathcal{F}_N$ happens with probability M_N, so

$$\mathbb{P}(W) = \mathbb{E}[\mathbb{E}[\mathbb{1}_W \mid R_N]] = \mathbb{E}[M_N] = 1/2.$$

□