Week 12: convergence in distribution

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 11/12/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1. Let $(X_i)_{i\geq 1}$ be a sequence of i.i.d. random variables following the uniform distribution on $[0,1]$.

(1) Show that $n\min(X_1,\ldots,X_n)$ converges in distribution to a random variable Z when $n\to\infty$ and give the law of Z.

(2) Show that

$$(X_1+\cdots+X_n)\min(X_1,\ldots,X_n) \quad \xrightarrow[n\to\infty]{(d)} \quad Z/2.$$

2 Training exercises

Exercise 2. Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables such that X_n has density p_n. Assume that there is a measurable function p such that $p_n(x)\to p(x)$ for λ almost all x (where λ is the Lebesgue measure).

(1) Is p always the density of some random variable? Justify your answer.

(2) Assume that there is an integrable (with respect to λ) measurable function $q:\mathbb{R}\to\mathbb{R}_+$ such that for every $n\geq 1$, $p_n(x)\leq q(x)$ for λ-almost all x. Show that p is the density of some random variable X and that X_n converges in distribution to X.

Exercise 3. Let $(X_n)_{n\geq 1}$ and X be real-valued random variables such that $\mathbb{P}(X=t)=0$ for every $t\in\mathbb{R}$. Show that X_n converges in distribution to X if and only if $\mathbb{P}(X_n<t)\to\mathbb{P}(X<t)=\mathbb{P}(X\leq t)$ for every $t\in\mathbb{R}$.

Exercise 4. Let $f:\mathbb{R}_+\to\mathbb{R}_+$ be a C^1 (continuously differentiable) weakly increasing function with $f(0)=0$.

(1) Let X be a non-negative real-valued random variable. Show that

$$\mathbb{E}[f(X)] = \int_0^\infty f'(x)\mathbb{P}(X > x)\mathrm{d}x.$$

(2) Let $(X_n)_{n\geq 1}$ be a sequence of non-negative real valued random variables converging in distribution to X.

(a) Show that $\mathbb{P}(X \geq 0) = 1$.

(b) Show that $\mathbb{E}[f(X)] \leq \liminf_{n\to\infty} \mathbb{E}[f(X_n)]$.

Exercise 5. Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables converging in distribution to a uniform random variable on $[0,1]$. Let $(Y_n)_{n\geq 1}$ be a sequence of real-valued random variables converging in probability to 0. Show that $\mathbb{P}(X_n < Y_n) \to 0$ as $n \to \infty$.

3 More involved exercise (optional, will not be covered in the exercise class)

Exercise 6. A stick of length 1 is broken at n points chosen uniformly and independently at random. Let L_n be the length of the longest of the $n+1$ pieces obtained. How does L_n behave when $n \to \infty$?

The aim of this exercise is to show that $(n+1)L_n - \ln(n+1)$ converges in distribution to a real-valued random variable whose cdf is $x \mapsto e^{e^{-x}}$ on $\mathbb{R}$ (called a Gumbel distribution).

Part 1. To model the problem, let $(U_i)_{1\leq i\leq n}$ be i.i.d. uniform random variables on $[0,1]$ representing the locations where the stick is broken.

(1) Show that $\mathbb{P}\Big(\exists i,j \in \{1,2,\ldots,n\} : i \neq j \text{ and } U_i = U_j\Big) = 0$.

(2) Show that there exists a random permutation σ such that $\mathbb{P}\Big(U_{\sigma(1)} < \cdots < U_{\sigma(n)}\Big) = 1$

Thus if $(\Delta_1,\ldots,\Delta_{n+1})$ denote the lenghts of the pieces, we have $\Delta_i = U_{\sigma_i} - U_{\sigma_{i-1}}$ for $1 \leq i \leq n+1$ (with the convention $U_{\sigma_{n+1}} = 1$ and $U_{\sigma_0} = 0$).

(3) Show that $(U_{\sigma(1)},\ldots,U_{\sigma(n)})$ has density

$$n!\, \mathbb{1}_{\{0\leq x_1<\ldots<x_n\leq 1\}}\, dx_1 \ldots dx_n.$$

Part 2. Let $(X_i)_{1\leq i\leq n+1}$ be exponential i.i.d. random variables with parameter 1. For $1 \leq i \leq n+1$, set

$$S_i = X_1 + \cdots + X_i, \qquad Y_i = \frac{X_i}{S_{n+1}}.$$

(4) Determine the joint law of $(X_1,\ldots,X_n,S_{n+1})$ and deduce that of $(Y_1,\ldots,Y_n)$.

(5) Show that $(\Delta_1,\ldots,\Delta_n)$ and $(Y_1,\ldots,Y_n)$ have the same distribution. Deduce that $\max(Y_1,\ldots,Y_{n+1})$ has the same law as L_n.

(6) Show that for $x \in \mathbb{R}$, $(x + \ln(n+1))\left(\frac{S_{n+1}}{n+1} - 1\right)$ converges in probability to 0.

(7) Deduce the desired result.

4 Fun exercise (optional, will not be covered in the exercise class)

Exercise 7. Let $n \geq 1$ be an integer. An urn contains n white balls and n colored balls. The balls are drawn successively and without replacement until there are only balls of one color left in the urn. As $n \to \infty$, what is the behavior of the number of remaining balls?