Week 13: characteristic functions, central limit theorem

Submission of solutions. Feedback can be given on Exercise 1 and any other exercise from the Training exercises. If you want to hand in, do it so by Monday 18/12/2023 17:00 (online) following the instructions on the course website

https://metaphor.ethz.ch/x/2023/hs/401-3601-00L/

Please pay attention to the quality, the precision and the presentation of your mathematical writing.

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1 Exercise covered during the exercise class

The following exercise will be covered during the exercise class.

Exercise 1.

(1) Let $(X_n)_{n\geq 1}$ be a sequence of real-valued random variables such that

$$\sqrt{n}(X_n - a) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0,\sigma^2)$$

with $a \in \mathbb{R}$ and $\sigma > 0$.

(a) Show that $X_n \to a$ in probability.

(b) Let $u : \mathbb{R} \to \mathbb{R}$ be a function such that $\lim_{x\to a} u(x) = 0$. Show that $\sqrt{n}(X_n - a)u(X_n) \to 0$ in probability.

Hint. First show that $u(X_n) \to 0$ in probability using the subsequence Lemma.

(c) Let $g : \mathbb{R} \to \mathbb{R}$ be a function such that it is differentiable at a with $g'(a) \neq 0$. Show that

$$\sqrt{n}(g(X_n) - g(a)) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, g'(a)^2 \cdot \sigma^2).$$

Hint. Using Taylor's expansion, write $g(x) = g(a) + (x-a)g'(a) + (x-a)u(x)$ with u a function having limit 0 at a.

(2) Fix $p \in (0,1]$ and for $n \geq 1$ let X_n be a $\mathsf{Bin}(n,p)$ random variable. Show that

$$\sqrt{n}\left(\ln\left(\frac{X_n}{n}\right) - \ln(p)\right) \xrightarrow[n\to\infty]{(d)} \mathcal{N}\left(0, \frac{1-p}{p}\right).$$

2 Training exercises

Exercise 2.

(1) Compute the characteristic function of an exponential random variable of parameter λ.

(2) Let $(X_i)_{1\leq i\leq n}$ be independent random variables such that X_i follows a Poisson(λ_i) distribution for every $1 \leq i \leq n$. Show that $X_1+\cdots+X_n$ follows a Poisson($\lambda_1+\lambda_2+\cdots+\lambda_n$) random variable.

(3) Find the limit of $e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!}$ as $n\to\infty$.

Hint. Use the central limit theorem.

Exercise 3. Let $(X_k)_{k\geq 1}$ be a sequence of i.i.d. standand $\mathcal{N}(0,1)$ random variables. Set

$$Y_n = \frac{1}{n}\sum_{k=1}^{n}\sqrt{k}X_k.$$

Study the convergence in distribution of Y_n.

Exercise 4. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. centered random variables with $\mathbb{E}[X_1^2]\in(0,\infty)$. Show that the sequence given by

$$Y_n = \frac{\sum_{k=1}^{n}X_k}{1+\left(\sum_{k=1}^{n}X_k^2\right)^{1/2}}$$

converges in distribution as $n\to\infty$ and identify its limit.

3 More involved exercises (optional, will not be covered in the exercise class)

Exercise 5. Let $(X_n)_{n\geq 1}$ be a sequence of i.i.d. real-valued random varirables. Assume that $\mathbb{E}[X_1^2]<\infty$. Set $m=\mathbb{E}[X_1]$, $\sigma^2=\mathrm{Var}(X_1)$ and $Z_n=\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(X_k-m)$.

(1) Recall the convergence in distribution of the sequence $(Z_n)_{n\geq 1}$.

(2) Show that $(Z_{2n}-Z_n)_{n\geq 1}$ converges in distribution and identity the limiting law.

Hint. Write $Z_{2n}-Z_n = aZ_n + bZ'_n$ for $a,b\in\mathbb{R}$ chosen in such a way that Z_n and Z'_n are independent and have the same law.

(3) Deduce that if $\sigma^2>0$, then the sequence $(Z_n)_{n\geq 1}$ does not converge in probability.

Remark. This shows that the convergence of the central limit theorem does not hold in probability.

Exercise 6. (Riemann-Lebesgue Lemma) Let X be a real-valued random variable having density p.

(1) Show that for every $\varepsilon>0$ there exists a simple function g of the form $\sum_i c_i \mathbb{1}_{A_i}$, where the A_i are open intervals of $\mathbb{R}$, such that $\int_{\mathbb{R}}|p(x)-g(x)|dx<\varepsilon$.

(2) Show that the characteristic function φ of X satisfies

$$\lim_{t\to\pm\infty} \varphi(t) = 0.$$

Exercise 7. Fix $\lambda > 1$ and let $(X_t)_{t\geq 0}$ be a family of random variables such that for every $t \geq 0$, X_t follows a geometric distribution with parameter $1 - e^{-t}$, that is

$$\mathbb{P}(X_t = k) = e^{-t}(1 - e^{-t})^{k-1}, \qquad k \geq 1.$$

Let $(U_n)_{n\geq 1}$ be a sequence of random variables such that $\lambda U_n - \ln(n)$ converges in probability to $-\ln(\mathcal{E})$ as $n \to \infty$, where $\mathcal{E}$ is an exponential random variable of parameter 1. Also assume that the two families $(X_t)_{t\geq 0}$ and $(U_n)_{n\geq 1}$ are independent.

Show that as $n \to \infty$, $X_{U_n}/n^{1/\lambda}$ converges in distribution to an exponential random variable, whose parameter is random and is equal to $\mathcal{E}^{1/\lambda}$.

4 Fun exercise (optional, will not be covered in the exercise class)

You have a box with n red balls and n blue balls. You take out each time a ball at random but, if the ball was red, you put it back in the box and take out a blue ball. If the ball was blue, you put it back in the box and take out a red ball.

You keep doing it until left only with balls of the same color. What is the behavior of the number of balls that will be left as $n \to \infty$?